COSMOLOGY

AN ELEMENTARY INTRODUCTION

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CONTENT

1. A brief introduction to modern astrophysics

1.1 Preamble

Astrophysics is a branch of science which has literally exploded during the past two or three decades, during which we have learnt more than what we knew previously. But at the same time it made us aware of the immensity of what remains to be understood. This is particularly true for cosmology, where it has become obvious that our ideas concerning gravity at large scales are very likely to require some revision: the discrepancy between the energy density of the Universe as we can measure it more or less directly and the value which we infer from our current ideas concerning the dynamics of the evolution of the expanding Universe ask for new physics and/or for a new theory of gravity at large distances. It is with this in mind that these notes have been written. Their aim is not to train future experts in general relativity but to open the students' minds to the main ideas that are at play in order to prepare them to the future.

Recently, astrophysics has attracted physicists from many different horizons: nuclear, particle, atomic, molecular, condensed matter, plasma and even life sciences. This diversity makes the field particularly attractive and dynamic. Students who wish to enter it today should get some familiarity with each of these branches of science before becoming too early specialized in a particular domain. In particular, some understanding of the ideas which govern our current thinking in cosmology, that is the study of the properties and evolution of the Universe on the largest possible scale, without paying attention to the details of its structure, is part of the knowledge which a student should acquire.

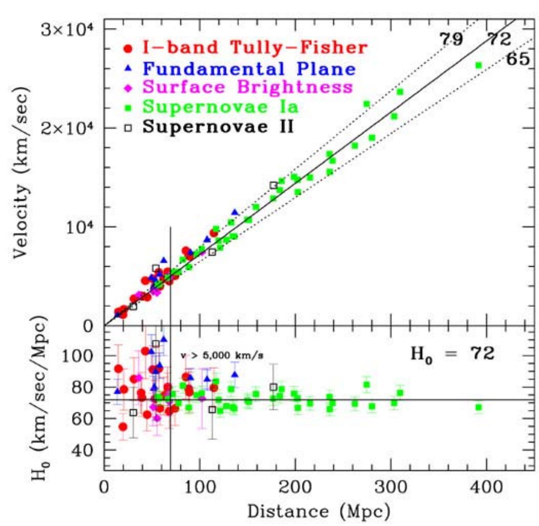


Figure 1.1. The Hubble velocity-distance relation

A number of coherent results have been obtained recently which have led to the construction of a standard model of astrophysics that is widely accepted in the community, the so-called concordance model. It is pictured in the framework of general relativity with the assumption that the Universe is isotropic and homogeneous at large scale, leading to the so-called Friedmann-Robertson-Walker (FRW) metric that embeds both the Hubble expansion of the

Universe and, locally, special relativity. Its study makes up much of the present notes. Within this model, presently available observations make it possible to measure the curvature of the Universe, which is found to cancel: the Universe is flat. Moreover, measuring the expansion of a flat Universe makes it possible to calculate its energy density. Recently, the addition of type Ia supernovae to the set of standard candles has allowed for major progress in the precision with which the Hubble (Figure 1.1) law has been verified: the Hubble constant is measured to be 71±3 km/s/Mpc. The main problem of cosmology today is that the energy density predicted in this way is four times as large as what we are able to account for. It is convenient to refer the energy densities that we are able to measure directly to this predicted density, defining ratios which are called Ω. Their sum, instead of being unity as expected, is only one quarter.

Evidence in favor of the big bang, or rather in favor of a state of very high temperature and density in which the Universe was, some 14 billion years ago, is overwhelming. Most impressive, and probably that which carries the most information, is the detection of a cosmic microwave background which is the remnant of the transition from an electron-nuclei plasma state to a state of neutral atoms which occurred when the Universe was only 400 000 years old. But other information is available that is also determinant, in particular the nucleosynthesis that occurred 2 to 3 minutes after the big bang, the observation of stars, galaxies and interstellar matter which is now being made over the whole electromagnetic spectrum, the direct observation of the expansion of the Universe from the Doppler red shift of recessing galaxies and the evidence for the presence of dark matter in the surroundings of galaxies and galaxy clusters.

1.2 Nucleosynthesis and direct observations

A few minutes after the big bang, the Universe had sufficiently expanded and cooled down to allow for nucleons, protons and neutrons, to form nuclei. Before that time, any nucleus that might have been formed would have immediately disintegrated into its nucleonic components. This process is referred to as nucleosynthesis or baryogenesis[1]. At that time the neutron to proton ratio was about 1/6, differing from unity because of the neutron-proton mass difference, $\Delta M=1.3$ MeV, (at equilibrium, the ratio of the two populations is given by the Boltzmann factor, $exp(-\Delta M/kT)$, where k is Boltzmann constant and T is temperature). While the Universe continued to expand, it quickly reached a state where the energy density was no longer large enough for protons and nuclei to overcome the Coulomb barrier: nuclear reactions could no longer take place.

[1] In principle, baryogenesis should be used to describe the confinement of quarks and gluons into hadrons (baryons or mesons) a microsecond or so after the big bang.

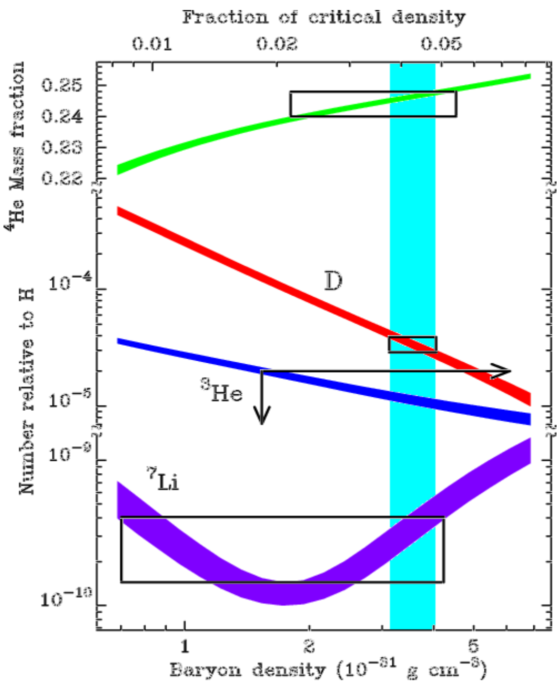


Figure 1.2 Nucleosynthesis, comparison of predictions and measurements of abundances *vs* density

As the neutron life time is 10 mn, neutron decays had no time to have a strong influence. The comparison between the size of the narrow window in temperature and pressure that allowed for nucleosynthesis to take place and the nuclear abundances which are measured today, severely constrains the parameters that described the Universe at that time, in particular its density and expansion rate. Of particular relevance to this analysis are the properties of light isotopes and the particularly strong binding of α particles, which are in a state of maximal spin and isospin symmetry. A corollary of this strong binding is that light even-even nuclei look like being made of α particles that are loosely bound together (so to speak, all the nuclear binding energy available has been used up to bind internally the α particles). In particular, the lightest possible even-even nucleus, ^{8}Be, is not stable and decays into two α particles as soon as it is formed. The cross-section to form heavier nuclei, in particular the next even-even nucleus, ^{12}C, is accordingly very low (it requires the simultaneous presence of three α particles in a same small volume or complicated catalytic reactions). Similarly, the direct formation of α particles from nucleons, requiring the simultaneous presence in a very small volume of four nucleons, is very unlikely. It must proceed via a chain of catalytic reactions called the p-p cycle (inside stars, burning hydrogen into helium may also proceed through the so-called CNO cycle but this was excluded at the time of nucleosynthesis). From accurate and reliable measurements of the abundances of helium and deuterium in the Universe, one finds (Figure 1.2) that the baryonic contribution to the predicted energy density of the Universe is only $\Omega_B = 4.4\%$. This is in fact four times larger than the contribution of stars as one can estimate it from direct observations in various domains of the electromagnetic spectrum, after extrapolation to the whole Universe. The progress made recently in our understanding of the evolution of stars, from birth to death, makes us believe that this result is reliable. It is only recently that one has identified the missing 3% as being mostly due to hot gases that have been detected in X ray astronomy and pervade many galaxy clusters. While a precise and foolproof estimate of Ω_B remains to be done, and will keep astrophysicists busy for many years to come, it seems difficult to imagine that one might be missing a major fraction of what is known today.

1.3 Dark matter

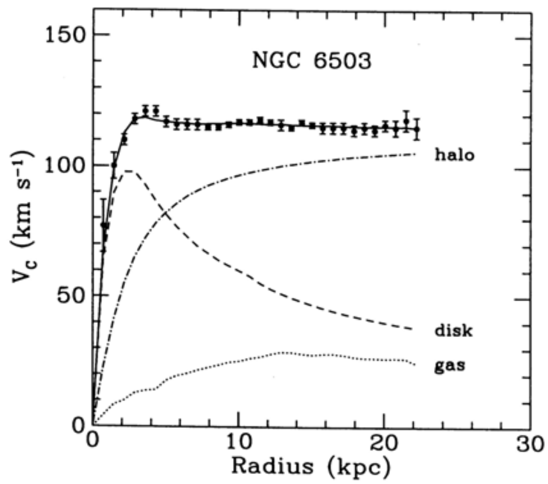


Figure 1.3 A typical velocity curve giving evidence for dark matter

The second component that has been extensively studied in the past two decades is dark matter. Evidence for it comes mostly from the velocity curves of stars far from the hub of spiral galaxies (Figure 1.3) (both the Milky Way and nearby galaxies) and from considerations on the binding of galaxy clusters. Velocity curves plot a star velocity V as a function of its distance r from the galactic centre. In a region far enough from the centre, one expects to be in the Kepler regime, namely all the attracting mass M is contained inside the orbit; in the approximation of a circular orbit, equilibrium implies $GM/r^2=V^2/r$, that is V should decrease with distance as $1/\surd r$. Instead, it remains constant deep into the region where the galaxy baryonic density has declined. Similarly, several clusters of galaxies should be unbound if they were to contain only baryonic matter. Detailed studies of these anomalies, together with other considerations such as the stability of typical spiral galaxies, have led to the conclusion that galaxies and galaxy clusters must be embedded into another form of matter that is called *dark matter*. It must be made of weakly interacting massive particles (*WIMP*) to explain what is observed. The need for massive particles (one speaks of *cold dark matter, CDM,* as opposed to *hot dark matter* in the case of relativistic particles) stems from the need to allow for the formation of galaxies in the early Universe, which would not have been possible with hot dark matter. The current estimate of the contribution of dark matter to the energy density of the Universe is $\Omega_{CDM} = 22.2\%$. Here again, this result is not considered to be controversial, even if the nature of dark matter is still a mystery. A natural candidate is the lightest supersymmetric partner of known particles, most probably a neutral spin ½ fermion. All efforts to find such particles have been defeated until now but a new accelerator under construction in Geneva, the *LHC*, should be able to produce and detect them, if they exist, within a few years.

1.4 Dark energy and the cosmic microwave background (CMB)

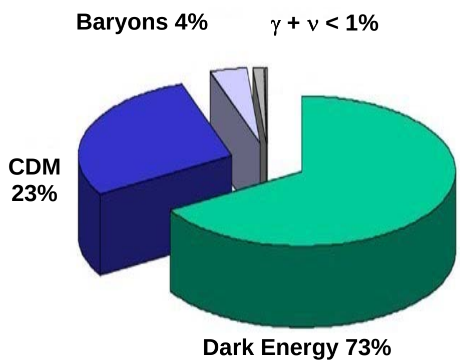


Figure 1.4 The content of the Universe

Baryonic matter and dark matter making up only 27% of the predicted density of the Universe, we are left with a deficit of 73% which is unexplained (Figure 1.4). It is best described by a cosmological constant, meaning a new[2] repulsive force that becomes important only at very large distances (that is distances commensurate with the size of the Universe, in the Gpc range). It would therefore be only recently, a few billion years ago, that the Universe became dark energy dominated and started accelerating its expansion rate rather than slowing it down as it was doing previously. Some additional evidence in favor of such a description is given by the observation of very large red shift galaxies being fainter than predicted. The question then remains to understand what a cosmological constant would mean physically.

However, other explanations need to be explored: our understanding of gravity at large distance may well be insufficient to allow for reliable predictions to be made: it would then be the prediction which is wrong, namely the predicted value of the energy density of the Universe. As this is largely obtained from CMB data (Figure 1.5), mostly from WMAP[3], it may be useful to very briefly describe the kind of information they carry:

– The Universe was extremely homogeneous and isotropic when the CMB observed today was emitted. This was at a red shift of 1000 or so from today, corresponding to the cooling down from plasma temperatures, in

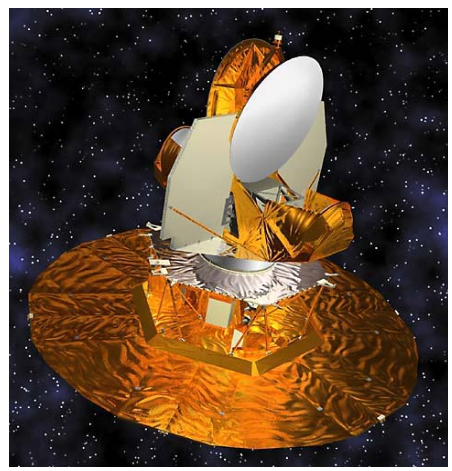


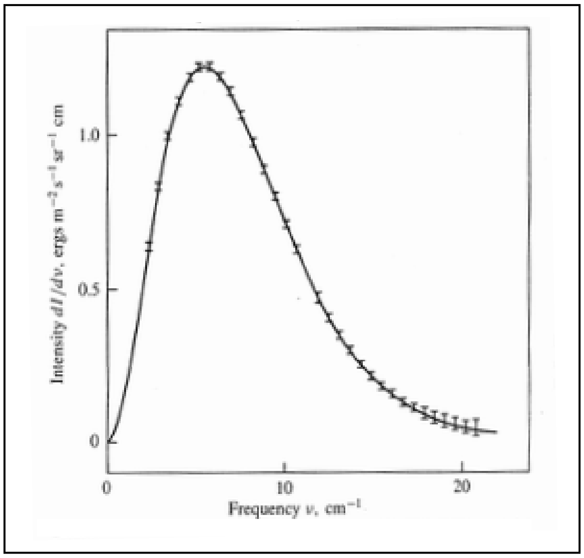
Figure 1.5 The Wilkinson Microwave Asymmetry Probe (WMAP)

[2] I mean a force which is neither part of Newtonian phenomenology nor part of the standard model of elementary particles.

[3] Wilkinson Microwave Asymmetry Probe.

the *eV* range, to the *2.7K* observed today. Since that time the Universe has been transparent and the CMB photons have traveled nearly undisturbed[4] until they reach us.

– The Universe was in perfect thermal equilibrium at that time as testified by the shape of the spectrum which agrees precisely with the Planck shape of a black body radiation spectrum (Figure 1.6).



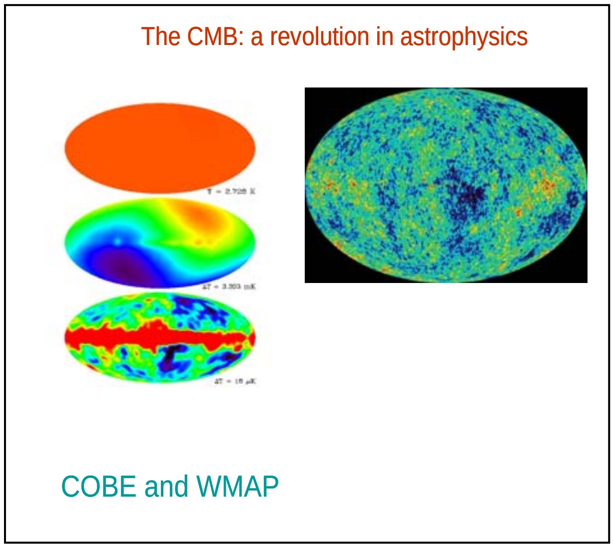


Figure 1.6 The black body Planck spectrum of the cosmic microwave background

Figure 1.7 Sky map of the CMB temperatures

– Below the 10^{-5} level or so, small inhomogeneities are observed in the temperature distribution over the whole sky (Figure 1.7). A Fourier analysis reveals that they are dominated by small spots of a size of approximately one square degree (Figure 1.8). These correspond exactly to the size of what was the horizon (namely the causally connected domain over which thermal equilibrium was precisely achieved) at the time of emission under the assumption of a flat Universe. A significant curvature would have resulted in a more diluted or more crowded pattern depending on its sign. The curvature of the Universe is measured this way by a quantity $k = -1 \pm 1.3\ \%$, a flat Universe having $k=0$.

The most important cosmological parameters (as given by the WMAP Collaboration in March 2006) are summarized in the table below. Each quantity is given as three numbers: the value, the positive uncertainty and the negative uncertainty.

[4] Except for a small perturbation at the time when galaxies formed, referred to as the reionization era, at a red shift of order 10. Information on this epoch can be extracted from a detailed study of the CMB power spectrum.

WMAP SUMMARY (March 2006)

Name	Symbol (unit)	Value	error +	error –
Hubble constant	$H(km/s/Mpc)$	70.9	2.4	3.2
Baryon fraction	Ω_B (%)	4.44	0.42	0.35
Matter fraction	Ω_{B+CDM} (%)	26.6	2.5	4.0
Critical density	$\rho_{crit}\,(10^{-26} kg/m^3)$	0.94	0.06	0.09
Dark energy	Ω_Λ (%)	73.2	4.0	2.5
Redshift reionization	z_{ion}	10.5	2.6	2.9
Age of Universe	$T(Gy)$	13.73	0.13	0.17
Equation of state	w	– 0.926	0.051	0.075
Spatial curvature	k	– 0.010	0.014	0.012

1.5 Mass and distance scales in the Universe

I close this brief introduction with Figure 1.9 that displays the positions of various physical objects in a mass *vs* distance plot. It is instructive in many respects. The units are the solar mass and centimeters. One should remember that the Schwarzschild radius is proportional to mass, that of the Sun being 3 km. The energy contained within the horizon of the evolving Universe, in both its matter and radiation dominated era, is proportional to its age. The Planck scale, $1/M_{Planck}$, is half the Schwarzschild radius associated to the Planck mass. White dwarfs and neutron stars fit between the Sun and Cyg X.

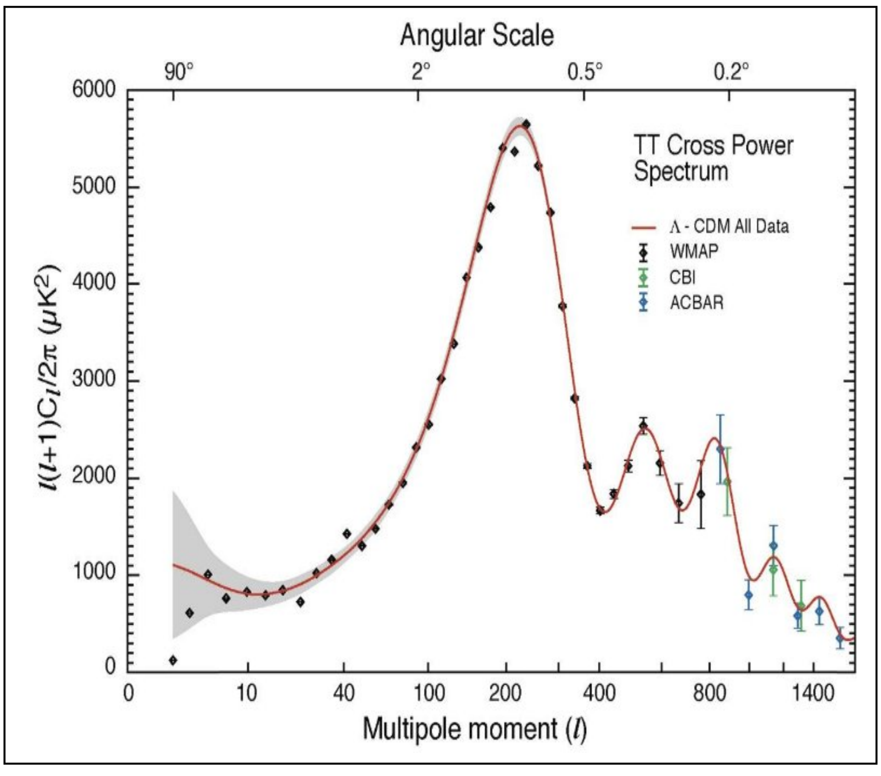
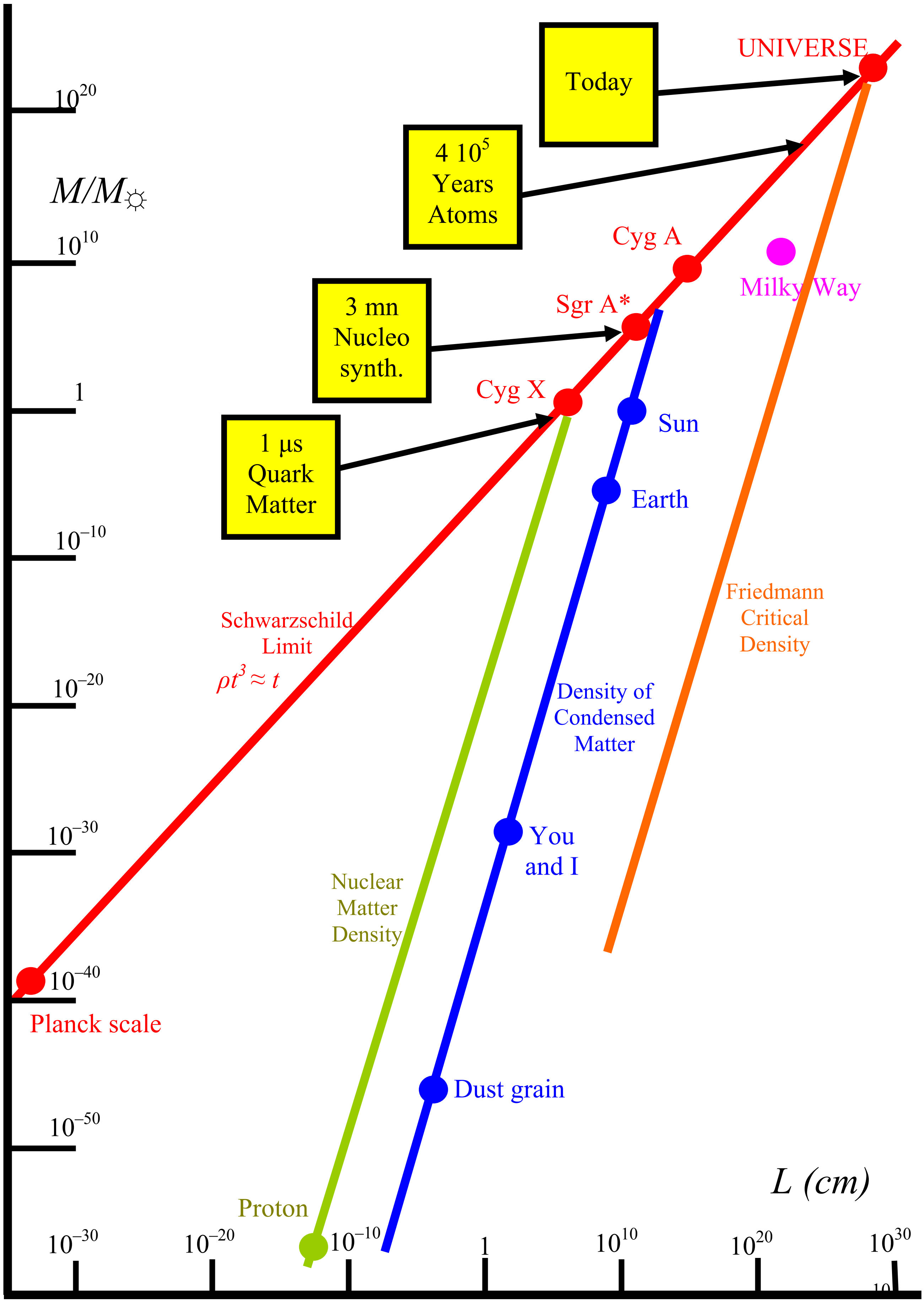


Figure 1.8 Fourier (or better spherical harmonic) analysis of the angular power spectrum of the CMB. The large peak is the first so called acoustic peak and is located precisely where one would expect it in the case of a flat Universe. The general decrease of the power spectrum toward higher multipole moments is consistent with inflation. Information is also contained in the peaks visible at high moments.



UNIVERSE
Today
4 10^5 Years Atoms
3 mn Nucleo synth.
1 µs Quark Matter
Cyg A
Sgr A*
Cyg X
Milky Way
Sun
Earth
M/M☼
Schwarzschild Limit
ρt³ ≈ t
Friedmann Critical Density
Density of Condensed Matter
You and I
Nuclear Matter Density
Planck scale
Dust grain
Proton
L (cm)
10^20
10^10
1
10^−10
10^−20
10^−30
10^−40
10^−50
10^−30
10^−20
10^−10
1
10^10
10^20
10^30

2. A brief reminder of special relativity

The lines below are only a reminder, the student is supposed to have learned the topic earlier. The accent is placed on aspects of relevance to cosmology.

Unless otherwise specified we use natural units where $\hbar = c = 1$.

2.1 Lorentz transformations

Lorentz transformations along *Ox* read (*y* and *z* being unchanged)

$x' = x \cosh\alpha + t \sinh\alpha$

$t' = x \sinh\alpha + t \cosh\alpha$

The system *S* in which *x* and *t* are measured moves along the *x* axis, which is the same as the *x'* axis, with velocity $\beta = \tanh\alpha$ measured in the system *S'* where *x'* and *t'* are measured.

In *S'* two events measured at a same time *t'* give

$(x_1 - x_2) \sinh\alpha + (t_1 - t_2) \cosh\alpha = 0$ and therefore

$x'_1 - x'_2 = (x_1 - x_2) \cosh\alpha + (t_1 - t_2) \sinh\alpha$

$= (x_1 - x_2) (\cosh\alpha - \sinh^2\alpha / \cosh\alpha)$

$= (x_1 - x_2) / \cosh\alpha.$

Namely **distances** appear to be **contracted** by a factor $\gamma = \cosh\alpha = 1/\sqrt{(1 - \beta^2)}$.

On the contrary, two events measured at a same location *x* in *S* give

$t'_1 - t'_2 = (t_1 - t_2) \cosh\alpha$. Namely **time differences** appear to be **dilated** by the same factor γ.

This result is not as trivial as it may sound, as it might seem to introduce an asymmetry between *S* and *S'*. Superficially, one might think that distances measured in *S* will appear dilated with respect to *S'* and that time differences measured in *S* will appear contracted, but it is not true of course. It is the measurement process that is not symmetric: to measure a distance in the fixed frame you compare two events that occur at the same time in the fixed frame while to measure a time difference in the fixed frame you compare two events that occur at the same location in the moving frame. It is important to have well understood this somewhat subtle difference.

As $\exp(\pm i\alpha) = \cos(\alpha) \pm i \sin(\alpha)$, $\cos(\alpha) = \{\exp(i\alpha) + \exp(-i\alpha)\}/2 = \cosh(i\alpha)$ and $\sin(\alpha) = \{\exp(i\alpha) - \exp(-i\alpha)\}/2i = -i \sinh(i\alpha)$. The Lorentz transformation may therefore be rewritten, replacing $\cosh\alpha$ by $\cos(-i\alpha)$ and $\sinh(\alpha)$ by $i \sin(-i\alpha)$

$x' = x \cos(i\alpha) - it \sin(i\alpha)$

$it' = x \sin(i\alpha) + it \cos(i\alpha)$

A Lorentz transformation is therefore a rotation by an angle $i\alpha$ in the (x, it) plane. In the same way as a rotation in the (x,y) plane leaves $x^2 + y^2$ invariant, the Lorentz transformation leaves $x^2 + (it)^2 = x^2 - t^2$ invariant. And in the same way as the rotation by an angle α simply increases the polar angle θ of the vector (x,y) by α, the Lorentz transformation increases by $i\alpha$ the equivalent of θ, namely $atan\ (it/x)=iargth(t/x)$. The quantity $argth\ (t/x)$, which increases by α in the Lorentz transformation, may also be written[5] $½ln\ \{(t+x)/(t–x)\}$. When referred to the energy-momentum four vector, this quantity is called *rapidity*, usually written y.

2.2 Addition of velocities

If a velocity $v_x=dx/dt$ parallel to Ox is measured in S, the velocity $v_x'=dx'/dt'$ measured in S' is $(dx\ cosh\alpha+dt sinh\alpha)/(dx\ sinh\alpha+dt\ cosh\alpha)=(v_x+\beta)/(1+\beta v_x)$. One recognizes here the law of addition of *tanh*, the product of two rotations being a rotation by the sum of the rotation angles. Whatever $v_x<1$ and $\beta<1$, v_x' is still a *tanh* and always smaller than 1: the light velocity cannot be exceeded by adding velocities that are themselves smaller than the light velocity. However one may conceive the existence of particles having velocities larger than the light velocity, such particles have received a name, *tachyons*, even though no evidence for them has ever been found. If a velocity $v_y=dy/dt$ normal to Ox is measured in S, the velocity $v_y'=dy'/dt'$ measured in S' is $dy/(dx\ sinh\alpha+dt\ cosh\alpha)=v_y/[\gamma(1+\beta v_x)]$.
Note that $v_y'/v_x'=v_y/[\gamma(v_x+\beta)]$ and $v'^2=v'^2_x+v'^2_y=(v_x^2+2\beta v_x+\beta^2+v_y^2–\beta^2v_y^2)/(1+\beta v_x)^2$. Writing $v^2=v_x^2+v_y^2$ we find $v'^2–1=(v^2–1)(1–\beta^2)/(1+\beta v_x)^2$. In particular, as $v'^2–1$ and $v^2–1$ have the same sign velocities smaller than the light velocity remain so and $v=1$ implies $v'=1$.

As an illustration of these relations let us consider a Hubble velocity field in a system S fixed with respect to a galaxy G_0 and ask ourselves how it would look like seen from a system S' fixed with respect to a galaxy G_1 at a distance a from G_0 (measured in G_0). Taking Ox along the line G_0G_1, we consider a third galaxy G having coordinates $(a+x, y)$ in G_0 and (x',y') in G_1. The velocity of G is $\{H(a+x), Hy\}$ in S, where H is the Hubble constant. The velocity of $G1$ in S is Ha and that of G_0 in S' is accordingly $–Ha$. At a distance $1/H$ from G_0 the velocity has reached the light velocity and beyond this horizon we have tachyons. We take G to be within this horizon. The velocity of G as seen from S' is
$\{(H[a+x]–Ha)/(1–H^2a[a+x]),Hy\sqrt{(1–H^2a^2)}/(1–H^2a[a+x])\}$
$=\{Hx/(1–H^2a[a+x]),Hy\sqrt{(1–H^2a^2)}/(1–H^2a[a+x])\}$.
This is different from what a Hubble field would look like in S', $\{Hx'=Hx\sqrt{(1–H^2a^2)},\ Hy'=Hy\}$. In S', not only the magnitude of the velocity does not obey the Hubble law but the velocity is no longer radial. Contrary to Euclidian geometry where the Hubble law is obeyed in any frame, the Hubble constant being

[5] Indeed $tanh(½ln\ \{(t+x)/(t–x)\})$
$=\{exp[½ln\{(t+x)/(t–x)]–exp[–½ln\{t+x)/(t–x)]\}/\{exp[½ln\{(t+x)/(t–x)]+exp[–½ln\{t+x)/(t–x)]\}$
$=\{\sqrt{(t+x)/(t–x)}–\sqrt{(t–x)/(t+x)}\}/\{\sqrt{(t+x)/(t–x)}+\sqrt{(t–x)/(t+x)}\}=(t+x–t+x)/(t+x+t–x)=x/t$.

the same at all points in space, it cannot be obeyed in special relativity. It can only be obeyed in a privileged frame and a Hubble field is not consistent with the cosmological principle. Of course, on the horizon, the velocity of G as seen from S' is equal to the light velocity: the horizon is the same in all frames: no new galaxy, which was behind the horizon, can be seen by moving closer to it.

Conversely, we may ask which velocity field, if there exists any, would be the same in any frame; namely does there exist a velocity field which is consistent with the cosmological principle. For reason of symmetry, we take the field to be radial in S, namely the velocity of G is $\{H(a+x),\ H(y)\}$ in S where H is now a function to be determined. The velocity of G in S' is

$\{[H(a+x)-H(a)]/[1-H(a)H(a+x)], H(y)\sqrt{[1-H^2(a)]}/[1-H(a)H(a+x)]\}$ which we want to be equal to $\{H(x\sqrt{[1-H^2(a)]}),H(y)\}$. As we know that the problem has no solution, let us be less exigent and look for a solution valid locally in the neighbourhood of $x=ka$. Developing $H(\varepsilon)=H_0\varepsilon+H_1\varepsilon^2+H_2\varepsilon^3$, we obtain for the x component

$$\{H_0ka+H_1(2+k)ka^2+H_2(3+3k+k^2)ka^3\}/\{1-H_0^2a^2(1+k)-H_0H_1a^3(1+k)(2+k)\}$$
$$=H_0ka+H_1(2+k)ka^2+H_2(3+3k+k^2)ka^3+H_0^3ka^3(1+k)$$
$$=H_0ka-½H_0^3ka^3+H_1k^2a^2+H_2k^3a^3$$

namely $H_1=0$ and $3H_2(1+k)=-(3/2+k)H_0^3$ or $H_2=-⅓H_0^3(3/2+k)/(1+k)$.

For the y component we find $\sqrt{[1-H^2(a)]}=1-H(a)H(a+x)$, namely $½H_0^2a^2+H_0H_1a^3=H_0^2a^2(1+k)+H_0H_1a^3(2+3k+k^2)$, that is $k=-1/2,\ H_2=-2/3\ H_0^3$.

This result, that there is no universal velocity field that satisfies the cosmological principle in special relativity, will shed some light on the Robertson-Walker metric when we come to study it (in that case, of course, the Hubble field will be consistent with the cosmological principle because, by then, we will have given up special relativity).

2.3 Energy and momentum, Maxwell equations

Before closing this section, let us recall that energy E and momentum $\boldsymbol{p}$ form a four vector, $E^2-\boldsymbol{p}^2=m^2$, m being the rest mass of the particle, a scalar. Its rapidity (measured along Ox), as was already said, is $y=½\ln\{(E+p_x)/(E-p_x)\}$. The leading terms of the development of E and $\boldsymbol{p}$ are $E=m+1/2mv^2$ and $\boldsymbol{p}=m\boldsymbol{v}$.

The gradient operator $\boldsymbol{\partial}$ transforms as $\partial_y=\partial_{y'}$, $\partial_z=\partial_{z'}$ and

$$\partial_{x'}=dx/dx'\ \partial_x+dt/dx'\ \partial_t=\cosh\alpha\ \partial_x-\sinh\alpha\ \partial_t$$
$$\partial_{t'}=dx/dt'\ \partial_x+dt/dt'\ \partial_t=-\sinh\alpha\ \partial_x+\cosh\alpha\ \partial_t$$

In vacuum, Maxwell equations read

$$\partial_t\boldsymbol{H}=-\boldsymbol{\partial}\times\boldsymbol{E}\quad \boldsymbol{\partial}.\boldsymbol{H}=0\quad \partial_t\boldsymbol{E}=+\boldsymbol{\partial}\times\boldsymbol{H}\quad \boldsymbol{\partial}.\boldsymbol{E}=0$$

In particular, $\partial_t E_x=\partial_yH_z-\partial_zH_y$ and $\partial_xE_x+\partial_yE_y+\partial_zE_z=0$

$$\partial_{t'}E_x=(-\sinh\alpha\ \partial_x+\cosh\alpha\ \partial_t)E_x=\sinh\alpha\ (\partial_yE_y+\partial_zE_z)+\cosh\alpha\ (\partial_yH_z-\partial_zH_y)$$
$$=\cosh\alpha\{\partial_y(H_z+\beta E_y)-\partial_z(H_y-\beta E_z)\}=\partial_{y'}(\cosh\alpha H_z+\sinh\alpha E_y)-\partial_{z'}(\cosh\alpha H_y-\sinh\alpha E_z)$$

The relativity principle implies that $\partial_{t'} E_{x'} = \partial_{y'} H_{z'} - \partial_{z'} H_{y'}$

Dealing similarly with the other equations, one obtains

$H_{x'}=H_x \quad E_{x'}=E_x$

$H_{y'}=cosh\alpha H_y - sinh\alpha E_z \quad E_{y'}=cosh\alpha E_y + sinh\alpha H_z$

$H_{z'}=cosh\alpha H_z + sinh\alpha E_y \quad E_{z'}=cosh\alpha E_z - sinh\alpha H_y$

These equations explicit the transformation of the electromagnetic field in vacuum. They show that the appearance of forces induced by the movement of a charge or of a current in a field is fully accounted for by the Lorentz transformation, namely the forces must be evaluated in the frame where the system is at rest. This result is a major success of the theory.

2.4 Accelerations

A fundamental difficulty of special relativity appears in the presence of accelerations. Consider two systems having the same origin and the same z and t axes but in uniform rotation with respect to each other around the z axis:

$x'=x\ cos\omega t - y\ sin\omega t, \quad y'=xsin\omega t+ycos\omega t$

In the (x,y) plane, at a distance r from the origin, there is a tangential velocity ωr and a radial acceleration $-\omega^2 r$. A particle cannot be at rest in that frame, otherwise at distance $1/\omega$ its velocity with respect to the other frame would reach the light velocity (note the analogy with our earlier discussion of a Hubble velocity field, here it is the tangential velocity that scales with r, in the Hubble case it was the radial velocity). The distance r is the same when measured in each of the two frames. But the tangential distance is contracted by a factor $\sqrt{(1-\omega^2 r^2)}$: for an observer in the fixed frame the ratio of the circumference of a circle to its radius is no longer 2π but $2\pi\sqrt{(1-\omega^2 r^2)}$. It will seem to this observer that space has been distorted. For $r=1/\omega$, namely when the tangential velocity reaches light velocity, the contraction factor cancels and the circumference reduces to a point. This is reminiscent of the geometry of a sphere of radius a: a parallel at $z=a\ cos\theta$ has a circumference $2\pi a\ sin\theta = 2\pi a\sqrt{(1-z^2/a^2)}$. Here a, the radius of curvature, plays the role of $1/\omega$. Accelerations seem therefore to imply a distortion of space.

Having noted the perfect symmetry of special relativity with respect to exchanges of coordinates (x, y, z, it) we may wonder what the rotation just considered corresponds to when changing coordinates. It is a rotation in the (x,y) plane by an angle $-i\omega it$. What about a rotation in the (z,it) plane by an angle $-i\omega x$, namely a Lorentz transformation along Oz with velocity $-\ tanh(\omega x)$? The equivalent of the circle is now a line $z^2-t^2=r^2$, the two branches of an hyperbola in the (z,t) plane which are scanned when x varies. This hyperbola is the same when viewed from both frames, but not the arc length along it. Between an event $x=y=t=0,\ z=r$ and an event $dx,dt,\ y=0,\ z=r$, both on the hyperbola, two observers will measure different arc lengths, that is different values of dt for a same dx. One of the observers may have *dx=dt,* namely the two events connected by a light ray, then the other observer will measure a different light velocity. From this point of view, accelerations seem to imply a non-constancy of the light velocity. The need to consider distortions of the metric properties of space and to abandon the hypothesis that the light velocity is a constant are the basic motivations for introducing general relativity as a theory able to handle frames accelerating with respect to each other.

2.5 Quantization

Special relativity can be quantized (at variance with general relativity), the equivalent of angular momentum corresponding to an imaginary spin operator. In the case of a spin ½ this operator has two eigenvalues, ±½, associated with two different subspaces of the Hilbert space, states of left handed and right handed particles respectively. The associated representation gives Dirac equation and implies the existence of particle and antiparticle states that transform into each other by charge conjugation. At the scale at which it can be tested, namely where gravity effects can essentially be neglected, special relativity has defeated all attempts at revealing deviations from what it predicts: it is verified to an extremely high precision.

3. *Extending the relativity principle to free falling frames: Gravity of photons*

3.1 Extension of the relativity principle to free falling frames

The idea that gravity can be described as a geometric property of space-time rather than as a dynamical process is at the root of general relativity. It makes an elegant use of the remark – which was made at Galileo time but which had not yet been made use of – that all masses fall in the same way in a gravity field. In Newtonian language this implies that the inertial and gravitational masses are equal[6] and are irrelevant to energy conservation: both the gravity potential and the kinetic energy are proportional to it. The extension of the relativity principle from inertial frames to free falling frames allows for describing locally, in any small space-time domain, the gravity field by an adequate acceleration given to the free-falling frame. Without going much further into the mathematics implied by these statements, one can deduce a host of important consequences touching the need for a revision of our concepts of space and time and for giving up special relativity, retaining it only locally. This is the subject of the present section. The general case, which leads to Einstein equations, will be treated next.

3.2 The action of gravity on photons, gravitational red shift

As a very simple illustration, consider a homogeneous gravity field directed along Oz in S. Let γ be the acceleration (Figure 3.1). In the free falling frame S' defined by the transformation $z'=z-1/2\gamma t^2$ all masses have a uniform linear movement. Extending the principle of relativity to this free falling frame we obtain a number of interesting results.

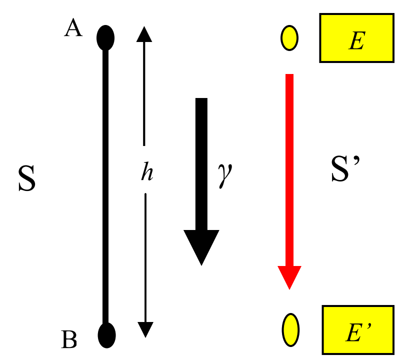


Figure 3.1 Gravity of photons

Take two points in S, A and B, on top of each other, A above B at a distance h of it. Send a photon of energy E from A to B. In B, the photon has an energy E', which would be equal to E if

[6] The first accurate measurement of the equality of the inertial and gravitational masses was due to Roland von Eötvös. It was later considerably improved by Robert H. Dicke and Vladimir B. Braginskii in the gravity field of the Sun and, more recently, by Eric G. Adelberger. The two masses are known to be equal to within 10^{-12}. A very precise analysis of the relative motion of the Moon with respect to the Earth, using laser reflectors left on the Moon by Apollo 11, 14 and 15 and Linakhod 2, has shown that the gravitational binding energy contributes identically to the inertial and gravitational masses.

the gravity field had no action on massless particles. To evaluate it consider the event in the free falling frame *S'* where the gravitational field vanishes. This system starts at zero velocity from *A* and reaches a velocity γt in *B*, with $t=h$ being the time it took for the photon to go from *A* to *B*. A photon being massless has equal energy and momentum, $E=p$. At *B*, the Lorentz transformation reads $E' = \cosh\alpha\, E + \sinh\alpha\, p$ where $\tanh\alpha = \gamma h$. To first order in α, $E'=E+E\gamma h$, namely the photon has acquired an additional energy $E\gamma h$ in the gravity field, corresponding to the usual $m_0\gamma h$ term in classical Newton mechanics, m_0 being the rest mass. It is indeed *E* and not m_0 that matters, it is energy that weighs, not rest mass[7]. Accordingly, when a star having a mass *M* and a radius *R* emits a photon of frequency ν, this photon is red shifted when it reaches far distances by an amount (remember that $E=\hbar\nu$) $\Delta\nu/\nu=\Delta E/E= \gamma R=GM/R$. One speaks of a gravitational red shift[8]. In the case of the Sun, the radius is 110 times larger than the Earth radius but the density is 4 times smaller, hence γ is 27 times larger, that is $0.27\ km/s^2$ and $\Delta\nu/\nu=0.27\times110\times6400/(3\ 10^5)^2=2\ 10^{-6}$. In the case of a neutron star $\Delta\nu/\nu$ may take values of order unity, in which case this first order estimate is no longer valid.

3.3 Schwarzschild metric and Birkhoff's theorem

Another way to look at the action of gravity on photons is to consider a mass *M* isolated in space, a star or a galaxy, and compare two different free falling frames: one has just enough velocity to escape to infinity, namely its metric is defined by the normal special relativity metric, $ds^2=dt^2-dl^2$; the other has less, enough to reach a distance *r* from *M*, at which point its velocity cancels and it falls back onto *M.* The timing is such that the first frame coincides with the second at the very moment where the latter has reached its turning point. The velocity *V* of the first frame at this moment is the escape velocity at *r*, such that $1/2V^2=MG/r$, that is $V=\sqrt{(2MG/r)}$ (we assume that *r* is large enough for *V* to be much smaller than *c* and Newtonian arithmetic to apply). The metric in the second frame is trivially obtained from that in the first frame by Lorentz transformation: distances are contracted and times dilated by a same factor, $1/\sqrt{(1-V^2)}=1/\sqrt{(1-2MG/r)}$. Hence

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[7] While m_0 was a scalar, *E* is not: gravity is not a scalar field. *E* is the fourth component of a four-vector, implying that gravity is in fact a tensor field: we will have to consider the energy-momentum tensor to describe what gravity couples to in the general case of a non uniform gravity field. At the end of the XIX[e] century, with the success of Maxwell equations, and even shortly after special relativity, many tried to describe gravity as a vector field but it had to fail. The carrier of gravity, the so-called graviton, has accordingly spin 2.

[8] Gravitational red shift on Earth (10^{-16} per meter!) has been measured using the Mössbauer effect by Robert V. Pound and his colleagues to an accuracy of one percent.
Using a hydrogen maser clock in a rocket at 10 000 km altitude, Robert F. C. Vessot and collaborators have measured the gravitational red shift to an accuracy of 2 parts in 10^4.

the metric in the second frame: $ds^2=(1-2MG/r)dt^2-(1-2MG/r)^{-1}dr^2$. This is called the Schwarzschild metric. Introducing the polar angles θ and φ, which are unaffected, it reads:

$$ds^2= (1-2MG/r)dt^2-(1-2MG/r)^{-1}dr^2-r^2(\sin^2\theta d\varphi^2+d\theta^2).$$

A singularity occurs at $R_{schwarzscild}=2MG$, the Schwarzschild radius, where the escape velocity is equal to the light velocity (equivalently, where a body falling from infinity, originally with zero velocity, has been accelerated to the light velocity). It corresponds to black holes.

The Schwarzschild metric, written here in the case of a single mass isolated in space, is in fact valid in a much more general case: Birkhoff has shown that Schwarzschild's metric holds in empty space surrounding any spherically symmetric mass distribution, even if this empty space is itself embedded in a larger, spherically symmetric distribution of matter.

3.4 Gravitational delay

Consider light traveling from the surface of the Sun to the Earth, namely $ds=d\theta=d\varphi=0$. Then $dt=dr/(1-2MG/r)$. The time t taken by the light to reach the Earth is therefore $\int dr/(1-2MG/r)=\int rdr/(r-2MG)=\int(u+2MG)du/u$ where $u=r-2MG$. Hence $t=t_0+2MG\ln\{(a_{earth}-2MG)/(R_{sun}-2MG)\}$ where t_0 is the time in the absence of gravity, a_{earth} is the radius of the Earth orbit and R_{sun} is the Sun radius. Putting numbers in gives a gravitational delay of the order of $50\ \mu s$, namely 10^{-7} times the uncorrected time. When sending a radar signal from the Earth to Venus and back, one may compare the extreme situations where Venus is on the other side or on the same side as the Earth with respect to the Sun, almost lined up in both cases. Then the difference in travel time corresponds to two full traversals near the Sun, namely $4\times50=200\ \mu s$. This has been verified with a precision of the order of a percent. The above calculation neglected the delay experienced by the photons when passing by the Sun because of their angular deviation. To estimate it, we set $ds=dr=d\theta=0$ and $\theta=\pi/2$ in the Schwarzschild metric. Hence, $dt=rd\varphi/(1-2MG/r)$. Putting numbers in, it corresponds to less than 10% of the gravitational delay calculated above.

3.5 Gravitational bending of light, gravitational lensing

The gravitational delay for a far away star seen near the Sun edge (Figure 3.2) is twice that given above for the gravitational delay from the Sun to the Earth, namely $\Delta t=4MG\ln(R-2MG)$ + terms that do not depend on R. Taking the derivative, we obtain $d(\Delta t)/dR \sim -4MG/R$, R being now the closest distance of approach to the Sun. This measures the angle by which the wave front planes are bent when

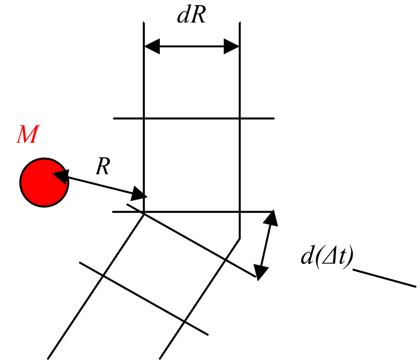
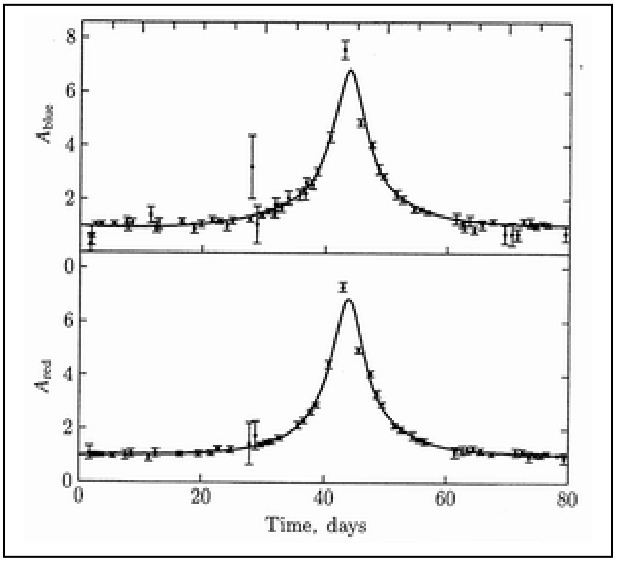


Figure 3.2 Gravitational lensing

passing near the Sun, which is also the angle by which the light rays are bent since they are normal to the wave fronts. It corresponds to nearly 2 seconds of arc.

The Sun may be thought of as being a weak lens with focal length equal to its radius divided by this angle of deflection, that is some 550 AU. Such gravitational lensing effects (Figure 3.3) are seen in many instances; in particular they produce so called *Einstein rings* on very distant quasars (Figure 3.4).



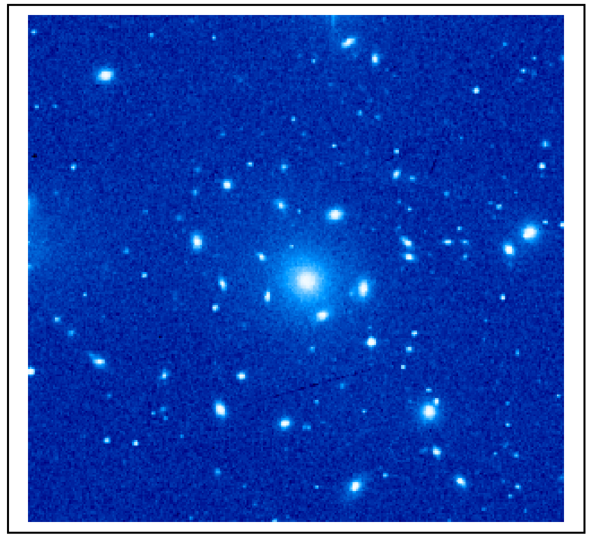


Figure 3.3 An illustration of gravitational lensing. What is shown here is the increase of luminosity of a LMC (Large MagellanicCloud) star in the background resulting from the passage in front of it of an obscure object in the halo of our galaxy. Observation is made both in the blue (top) and in the red (bottom).

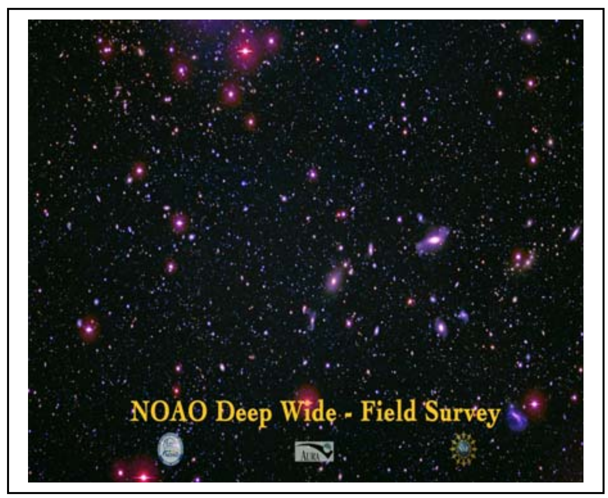


Figure 3.4 Deep field surveys giving evidence for very old galaxies with redshifts reaching 10 and showing several examples of Einstein rings.

4. General relativity and Einstein equations

The results of the last chapter have shown that the extension of the relativity principle to free falling frames implies a violation of special relativity and a serious revision of our concepts of space and time. In particular, our discussion of gravitational delay has shown that the light velocity can no longer be considered as a constant, a gravitational correction of order $2MG/r$ must be applied. While the relativity principle is not only still valid but even extended to free falling frames, the constancy of the light velocity, the other basic assumption of special relativity, must now be abandoned.

4.1 The strategy

In formulating general relativity, Einstein was very attentive at extending the relativity principle to as many frames as possible. The idea was to start from the local frame – x_μ , having a metric $g^{\mu\nu}$ – in which there is a gravity field and transform to a local free falling frame X_μ where special relativity applies. The X_μ frame would therefore have a metric $G^{\mu\nu}=(-1,\ -1,\ -1,\ 1)\delta^{\mu\nu}$. The transformation would be linear and read $X_\mu=\Omega_\mu{}^\nu x_\nu$: it is sensible to only consider homogeneous transformations as the effect of translations is trivial and only complicates the writing. However, Einstein could not take it as granted that such a transformation exists. Indeed, if there is a point mass at the origin, no transformation can get rid of it. Only in vacuum, where there is no matter and no radiation is it reasonable to take it as granted that such a transformation should exist.

Before pursuing, let us have a closer look at the metric and how it transforms. The metric tensor (Einstein calls it the fundamental tensor) is defined from the relation below

$$ds^2=g^{\alpha\beta}dx_\alpha dx_\beta \tag{4.1}$$

Writing $ds^2=dS^2=G^{\mu\nu}dX_\mu dX_\nu=G^{\mu\nu}(x_\sigma\partial^\alpha\Omega_\mu{}^\sigma+\Omega_\mu{}^\alpha)dx_\alpha(x_\tau\partial^\beta\Omega_\nu{}^\tau+\Omega_\nu{}^\beta)dx_\beta$ with $\partial^\alpha=\partial/\partial x_\alpha$ we get $g^{\alpha\beta}=G^{\mu\nu}(x_\sigma\partial^\alpha\Omega_\mu{}^\sigma+\Omega_\mu{}^\alpha)(x_\tau\partial^\beta\Omega_\mu{}^\tau+\Omega_\mu{}^\beta)$. This metric depends on the coordinates in the general case. Hence, while a geodesic is a straight line in the system X_μ it is in general a curve in the system x_μ. It is only when the $\Omega_\mu{}^\nu$ do not depend on coordinates that the $g^{\mu\nu}$ don't either. But in that case a straight line transforms into a straight line and the x_μ system is an inertial frame in the sense of special relativity. The $g^{\mu\nu}$, in some sense, are the coordinates of the gravity field in the x_μ system: they describe it completely. How one goes from 10 $g^{\mu\nu}$ (10 and not 16 as the metric tensor is by definition symmetric) to the components of the gravity field does not need to be explicitly considered here.

Coming back to Einstein's strategy, it consists therefore in considering a space with an arbitrary metric $g^{\mu\nu}$ and start by asking which condition this metric must obey for a transformation to a free falling frame to exist; that is to an inertial frame in the sense of special relativity, namely a frame having the metric $G^{\mu\nu}$, or, more generally, a frame having a metric that does not depend on the coordinates. The only restriction Einstein places on the metric $g^{\mu\nu}$ is that $det(g^{\mu\nu})=det(G^{\mu\nu})= -1$. We take it as always obeyed from now on. Having established these conditions, which are in some sense the field equations in vacuum, Einstein studied which function of the metric was representing the gravitational energy and, having found it, he wrote the general field equations by simply adding to the gravitational energy in vacuum that contained in matter and radiation. Namely, in his formulation, one essentially does not talk about gravity: the whole physics is contained in the metric and its relation to the energy carried by whatever matter and radiation there is around. From a physics point of view, knowing the metric is sufficient because the equations of movement are obtained by finding the geodesic, namely by minimizing $\int ds$ between two points 1 and 2: the whole of physics is indeed contained in the metric.

4.2 Field equations in vacuum

Einstein therefore started from a space having a metric obeying the determinant condition and asked which other condition had to be obeyed for being able to transform into a frame having a metric that does not depend on coordinates.

At this point it is useful to recall two theorems of differential geometry that address such questions.

Theorem 1 solves the problem of finding the geodesic given the metric. The equations of movement read

$$d^2x_\tau/ds^2=\Gamma^{\mu\nu}{}_\tau\, dx_\mu/ds\; dx_\nu/ds \qquad (4.2a)$$

with the Christoffel symbols $\Gamma^{\mu\nu}{}_\tau$ given by

$$\Gamma^{\mu\nu}{}_\tau = ½\, g^{\tau\alpha}\, (\partial^\nu g_{\mu\alpha}+\partial^\mu g_{\nu\alpha}-\partial^\alpha g_{\mu\nu}) \qquad (4.2b)$$

Theorem 2 states that the condition for a frame to be transformable into another frame where the metric tensor does not depend on the coordinates, is that all components of the Riemann tensor $R_{\mu\nu}$ cancel, $R_{\mu\nu}$ being defined as

$$R_{\mu\nu} = \partial^\alpha\Gamma^{\mu\nu}{}_\alpha+\Gamma^{\mu\alpha}{}_\beta\Gamma^{\nu\beta}{}_\alpha \qquad (4.3)$$

This condition is a condition placed on the metric.

It is now easy to write the conditions which the metric must obey in vacuum, that is far from any matter or radiation. In that case, one must be able to transform to a frame where the metric is constant, therefore the Riemann tensor must cancel and one obtains the 10 equations that constrain the metric tensor, which are in some sense the field equations in vacuum:

$$\partial^{\alpha}\Gamma^{\mu\nu}{}_{\alpha}+\Gamma^{\mu\alpha}{}_{\beta}\Gamma^{\nu\beta}{}_{\alpha}=0 \tag{4.4}$$

Even far from any matter or radiation, there is gravitational energy: how does it relate to the metric? At this point a third theorem is useful:

Theorem 3 introduces the Hamiltonian $H= g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha}$ and, noting that the equations (4.4) are equivalent to solving $\delta\int Hdx_{\tau}=0$, states that the quantities $t^{\alpha}{}_{\sigma}$ defined below are invariants

$$\begin{gathered}\partial^{\alpha}t^{\alpha}{}_{\sigma}=0\\ \kappa t^{\alpha}{}_{\sigma} = -\tfrac{1}{2}\{\partial^{\sigma}g^{\mu\nu}\partial H/\partial(\partial^{\alpha} g^{\mu\nu})-\delta^{\alpha}{}_{\sigma}H\} = \tfrac{1}{2}\delta^{\alpha}{}_{\sigma}g^{\mu\nu}\Gamma^{\lambda}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\lambda}- g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\sigma}\\ H= g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha},\quad \kappa=8\pi G\end{gathered} \tag{4.5}$$

It is then possible to rewrite equations (4.4) with the $t^{\alpha}{}_{\sigma}$ appearing explicitly, with the result below (writing $t=\sum_{\alpha} t^{\alpha}{}_{\alpha}$)

$$\partial^{\alpha} (g^{\lambda\nu}\Gamma^{\alpha}{}_{\mu\nu}) = -\kappa\, (t^{\lambda}{}_{\mu}- \tfrac{1}{2}\delta^{\lambda}{}_{\mu}t) \tag{4.6}$$

This relation relates explicitly the metric in the left hand side to the energy content of the gravity field in vacuum on the right hand side. Indeed, the $t^{\lambda}{}_{\mu}$ may be thought of as the energy components of the field, their conservation (Relation 4.5) expressing energy-momentum conservation. The coefficient $\kappa=8\pi G$ that has been introduced gives them the proper scale as we shall see below when considering the Newton approximation. While equations 4.4, 4.5 and 4.6 are equivalent forms of the field equations in vacuum, the latter is written in a form which invites one to extend it to the case where there is matter and/or radiation around.

4.3 Einstein equations

Starting from equations 4.6 Einstein then postulated that in the general case, namely in the presence of matter and radiation, the field equations can be simply written by replacing $t^{\lambda}{}_{\mu}$ by the sum $(t^{\lambda}{}_{\mu}+T^{\lambda}{}_{\mu})$ where $T^{\lambda}{}_{\mu}$ is the energy-momentum tensor of the matter and radiation, with trace T. One obtains this way Equations

4.6a. It is then simple to work one's way backward to obtain Equation 4.4a, the equivalent of Equation 4.4[9]

$$\partial^\alpha\,(g^{\lambda\nu}\Gamma^\alpha{}_{\mu\nu}) = -\,\kappa\,\{(t^\lambda{}_\mu+T^\lambda{}_\mu) - \tfrac{1}{2}\delta^\lambda{}_\mu(t+T)\} \qquad (4.6a)$$
$$\partial^\alpha\Gamma^{\mu\nu}{}_\alpha+\Gamma^{\mu\alpha}{}_\beta\Gamma^{\nu\beta}{}_\alpha = -\,\kappa\,(T^{\mu\nu} - \tfrac{1}{2}\delta^{\mu\nu}T) \qquad (4.4a)$$

Equations 4.4a and 4.6a are two equivalent forms of Einstein equations. They relate the metric to the energy momentum tensor of whatever matter and radiation there is around. The movement can then be deduced from the metric *via* Equations 4.2. In no place does one need to talk about gravity: its concept has been completely absorbed into that of space.

An essential justification of the procedure used by Einstein is that it naturally leads to energy-momentum conservation in the general case, namely Equation 4.5 now becomes

$$\partial^\alpha(t^\alpha{}_\sigma+T^\alpha{}_\sigma) =0 \qquad (4.5a)$$

implying

$$\partial^\alpha T^\alpha{}_\sigma = -\,\Gamma^\alpha{}_{\sigma\beta}T^\beta{}_\alpha \qquad (4.5b)$$

Here, the right hand side describes explicitly the action of the gravity field on the energy-momentum of the matter and radiation present.

4.4 Newton approximation

In the limit of low velocities (with respect to light velocity) dx_4/ds is equal to 1 to second order in the dx_i/ds $(i=1$ to $3)$. In (4.2a) the equations of movement are therefore dominated by $d^2x_\tau/ds^2=\Gamma^{44}{}_\tau\,(dx_4/ds)^2$ which, to leading order, becomes $d^2x_\tau/dx_4{}^2=\Gamma^{44}{}_\tau$. But $\Gamma^{44}{}_\tau = \tfrac{1}{2}\ g^{\tau\alpha}\ (\partial^4 g_{4\alpha}+\partial^4 g_{4\alpha}-\partial^\alpha g_{44})$; in this expression the derivatives with respect to time can be neglected when compared to those with respect to space and $\Gamma^{44}\tau = -\,\tfrac{1}{2}\,\partial^\tau g_{44}$, hence, writing $t=x_4$, $d^2x_\tau/dt^2= -\,\tfrac{1}{2}\,\partial g_{44}/\partial x_\tau$. This is the result of Newton theory, $\tfrac{1}{2}\ g_{44}$ being identified with the gravity potential Φ.

Concerning the field equations $\partial^\alpha\Gamma^{\mu\nu}{}_\alpha+\Gamma^{\mu\alpha}{}_\beta\Gamma^{\nu\beta}{}_\alpha= -\,\kappa\,(T^{\mu\nu} - \tfrac{1}{2}\delta^{\mu\nu}T)$ (4a), let us recall the expression of the momentum energy tensor as a function of pressure p and density ρ:

$$T^{\alpha\beta}= -g^{\alpha\beta}\,p+\rho dx_\alpha/ds\ dx_\beta/ds \qquad (4.7)$$

[9] We ignore, in the present chapter, the possible addition of a cosmological constant Λ that would imply a modified Relation 4.4a : $\partial^\alpha\Gamma^{\mu\nu}{}_\alpha+\Gamma^{\mu\alpha}{}_\beta\Gamma^{\nu\beta}{}_\alpha=\Lambda g_{\mu\nu} - \kappa\,(T^{\mu\nu} - \tfrac{1}{2}\delta^{\mu\nu}T)$. We shall return to this topic in Section 6.2.

Limiting the discussion to cases where the contribution of p is negligible, we find again to leading order that all coordinates cancel apart for $T^{44}=\rho$. The trace is therefore also equal to ρ. Replacing in 4.4a, $\partial^{\alpha}\Gamma^{44}{}_{\alpha}+\Gamma^{4\alpha}{}_{\beta}\Gamma^{4\beta}{}_{\alpha}= - \kappa\rho/2$ and, neglecting as before the derivatives with respect to time, $\partial^{\alpha}\partial^{\alpha}g_{44}=\kappa\rho$ which is Newton's law, $\Delta g_{44}=8\pi G\rho$, namely $\Delta\Phi=4\pi G\rho$. This justifies the normalization chosen in the definition of κ.

5. A homogeneous and isotropic Universe
Robertson-Walker metric and Friedman equations

5.1 The cosmological principle

The evidence that the Universe was isotropic some 400 000 years after the big bang, obtained from the uniformity of the CMB temperature over the sky, is quite good. The accuracy is at a few per mil level, limited by our precise knowledge of the movement of the solar system with respect to the CMB. In fact COBE and WMAP data have been used to infer this movement and, after subtraction, the uniformity of the CMB is at the few 10^{-6} level; but one might argue that there exists an asymmetry of the Universe conspiring with this movement. Also, the counting of galaxies gives a uniform result across the sky, even if the presence of the Milky Way prevents a full survey to be made. Here, one has to be careful to account for distance; otherwise one could only conclude that the Universe is locally isotropic. A valuable statement can only rely on the survey of very distant galaxies.

Of course, we are only talking here about the Universe at very large distances, we know pretty well, just looking around us, that the Universe is far from looking as a homogeneous and isotropic fluid. How far do we have to go? in fact one finds large scale structures nearly up to the horizon, namely on the Gpc scale, with walls, filaments and voids.

It is sometimes said and written, erroneously, that, since Copernicus time, we know that there is no privileged place in the Universe. This is wrong of course. What we know, is that we, the Earth, the solar system, the Milky Way, do not occupy a privileged place in the Universe. But the Sun surely does in the solar system, and SgrA* surely does in the Milky Way.

A serious limitation on our ability to judge of the isotropy and homogeneity of the Universe is the fact that our view of it is only partial: we do not know what is going on behind the horizon. In fact the precise meaning that should be given to this sentence depends on the cosmological model we have in mind, we'll come back to this later. Current ideas about the very early inflatory evolution of the Universe imply that we can see only a minute fraction of it. It is therefore conceivable that there exists some centre of the Universe some 100 or so horizons away from us or, if you do not like the idea of a centre for some philosophical reason, the Universe might have some kind of periodic structure on a similar scale: we would not have noticed it. I am not claiming that these are hypotheses to consider seriously, I am just protecting the students from too blind an adhesion to the official dogma.

Moreover, it is clear that there does exist a privileged frame, that in which, on average, all galaxies are at rest. This statement should not be disposed of too lightly; it is in fact quite troublesome and not well understood. It has fed the reflection of many cosmologists, such as Einstein and Wheeler, without ever finding a satisfactory answer. It is sometimes referred to as Mach's principle (but it is in fact Einstein who gave it this name). The simplest illustration is to think of two elastic bodies having cylindrical symmetries around a same *z* axis, in rotation with respect to each other, far from each other and far from any other body. One of these is flattened along the *z* axis (stretched normally to it) and the other is not. We would say that the first one is distorted under the influence of the centrifugal force and that the other is at rest. But there is no way to privilege one frame with respect to the other unless we refer them to something else, something external to the system, in that case very distant galaxies. I am afraid that if one is not troubled by this statement, one has not really understood it. It implies that the inertial properties of the bodies considered here depend on the gravitational action of distant galaxies; it is closely related to the equality of the inertial and gravitational masses. It also implies that we cannot think of a space containing only the two bodies in question, namely one cannot ignore the rest of the Universe. If one could the principle of relativity would be violated unless the two fluids would behave in the same way whatever their relative rotation. General relativity forces us to think of space as generated by what it contains, not as a frame that might be empty of energy. It surely sheds a new light on Mach's principle but one cannot say that it fully answers the questions raised.

Anyhow, it is generally assumed, as a working hypothesis, that the Universe is isotropic and homogeneous. There is surely no evidence to the contrary, at least locally at the Gpc scale, but it is important not to erect this statement into a dogma and to keep one's eyes open for possible deviations. This hypothesis is usually referred to as the cosmological principle, a very bad name indeed. It is just one hypothesis among others, and one of the least solid.

5.2 The Friedmann-Robertson-Walker (FRW) metric

Under such an assumption the energy momentum tensor takes a remarkably simple form. As the pressure and energy density must be independent of space, there are just two time dependent quantities that describe the Universe. Through Einstein equations, this implies that, at any time, its metric is completely defined by two numbers. On the basis of group-theoretical arguments, independently, Robertson and Walker have given its form as

$$ds^2=dt^2-dl^2 \text{ where } dl^2=a^2(t)\{d\chi^2+\sigma^2(\chi)[d\theta^2+\sin^2\theta d\varphi^2]\}$$

and Einstein equations take the form of Friedmann equations that relate the energy density and the pressure to da/adt and d^2a/adt^2.

Before pursuing, let us study the metric of simple curved spaces. In the same way as we can think of a sphere in a three dimensional space as a curved two dimensional surface, we can think of a hypersphere in a four dimensional space as a curved three dimensional volume. Its equation reads $x_1^2+x_2^2+x_3^2+x_4^2=\pm a^2$. Here the $\pm$ sign is to include hyperhyperboloids as well as hyperspheres in our discussion. Writing $r^2=x_1^2+x_2^2+x_3^2$, $x_4^2=\pm a^2-r^2$ but, on the hypervolume, $rdr+x_4dx_4=0$. Hence $dx_4^2=(rdr)^2/x_4^2=(rdr)^2/(\pm a^2-r^2)$ and, using polar coordinates in the space (x_1, x_2, x_3), $dl^2=dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2+r^2dr^2/(\pm a^2-r^2)$. It is convenient to write $r=a\sigma(\chi)$ with $\sigma(\chi)=\sin\chi$ for the + sign and $\sinh\chi$ for the – sign.
Then $dr^2+r^2dr^2/(\pm a^2-r^2)=dr^2(\pm a^2)/(\pm a^2-r^2)=a^2d\chi^2\cos^2\chi\, a^2/(a^2\cos^2\chi)=a^2d\chi^2$ in the + case and $=a^2d\chi^2\cosh^2\chi(-a^2)(-a^2\cosh^2\chi)=a^2d\chi^2$ in the – case. Hence $dl^2=a^2\{d\chi^2+\sigma^2(\chi)[d\theta^2+\sin^2\theta d\varphi^2]\}$. In case of a flat space the metric is simply $dr^2+r^2[d\theta^2+\sin^2\theta d\varphi^2]$: a is an irrelevant scale and, calling $r=\chi$, $\sigma(\chi)=\chi$.

We may therefore write the metric as either of two ways

$$dl_a^2=a^2\{dr^2/(1-kr^2)+r^2[d\theta^2+\sin^2\theta d\varphi^2]\} \quad (5.1a)$$
$$dl_a^2=a^2\{d\chi^2+\sigma^2(\chi)[d\theta^2+\sin^2\theta d\varphi^2]\} \quad (5.1b)$$

where a defines the scale. One must distinguish between three cases: $k=0$ and $\sigma(\chi)=\chi$ corresponds to a flat space; $k=1$ and $\sigma(\chi)=\sin\chi$ corresponds to a closed space; $k=-1$ and $\sigma(\chi)=\sinh\chi$ corresponds to an open space. Parameter k is the Riemann curvature of the space.

The FRW metric is simply

$$ds^2=dt^2-dl^2_{a(t)} \quad (5.2)$$

namely the metric of special relativity that has been extended by accepting a possible space curvature (three possibilities) and introducing a space dependent scale $a(t)$.

An object having fixed χ, θ and φ coordinates has its movements fully defined by the time dependence of the scale $a(t)$. Its x_1, x_2 and x_3 coordinates at time t are simply what they were at time t_0 multiplied by the scale factor $a(t)/a(t_0)$. One speaks of a comoving frame and of comoving coordinates. In a comoving frame, stars and galaxies are usually not at rest but have proper movements, sometimes called peculiar movements, with relative velocities with respect to each other at the typical scale of a permil of the light velocity. At such a level relativistic corrections are small and, for many purposes, can be ignored. The time dependence of the metric, here of $a(t)$, is – as we have seen when studying Einstein equations – related to the gravity field. It is a linear scaling, expansion or contraction. Neglecting peculiar movements, light traveling from a galaxy to another is simply obtained by writing $ds=0$ in the FRW metric, namely, taking one galaxy at the origin of coordinates, and noting that the other galaxy moves along the line of sight $d\theta=d\varphi=0$, we find $dt=dl_{a(t)}=a(t)d\chi$ giving

$$\chi=\int dt/a(t) \tag{5.3}$$

where the integral is from the time of emission of the light signal to its time of reception (measured by an observer having fixed comoving coordinates, a so-called fundamental observer, one speaks of world time). χ is called the distance parameter, it is the time taken by light to go from one galaxy to another, both of them being fixed in the comoving frame. The distance $a(t_0)\sigma(\chi)$, namely the distance measured in the comoving frame at time t_0, is called the cosmological distance.

Let galaxy G_0 emit a signal of frequency ν_0 between world time t_0 and world time t_0+dt_0 and galaxy G_1 receive the signal as having frequency ν_1 between world time t_1 and world time t_1+dt_1. Obviously $\nu_0\,dt_0=\nu_1\,dt_1$ (the number of clock beats is conserved). In terms of wavelength $\lambda=1/\nu$, we find

$$z=(\lambda_0-\lambda_1)/\lambda_1=dt_0/dt_1-1=a(t_0)/a(t_1)-1 \tag{5.4}$$

This important relation defines the measured red shift parameter z. We see that $1+z$ is a direct measure of the expansion.

Each time one talks about a distance, one has to specify what is meant, namely what is measured. An important relation is between distance and apparent luminosity. The apparent luminosity is proportional to

$$L_0\,[a_0\,\sigma(\chi)]^{-2}[1+z]^{-2} \tag{5.5}$$

where L_0 is the luminosity at time of emission. The first factor represents the dilution of radiation, namely the decrease in solid angle; in the second red shift factor one $(1+z)$ accounts for the decrease in energy per photon received, the other for the lesser number of photons received per unit of time, a consequence of the apparent slowing down of clocks.

5.3 Friedmann equations

As the evolution of a homogeneous and isotropic Universe is fully described by the time dependence of its scale (there are no peculiar movements), we should not be too surprised that an elementary Newtonian treatment of the problem gives the same result as a rigorous application of Einstein equations. This is the road that I chose to follow here, in the hope that, from a didactic point of view, it will leave a very simple picture in the mind of the students. What follows in this section can be understood by anyone having some very rudimentary knowledge of physics. I hope that, when it is read in the light of what precedes, the offence it makes to rigor can be excused.

A well known property of any field of force inversely proportional to distance[10] is that in a spherically symmetric medium (such a medium has therefore a centre, O) the force acting on a mass at distance r from O is directed toward O and is the same as if all the mass contained inside the sphere of centre O and radius r was concentrated in O. Namely the action of all matter external to this sphere cancels. Birkhoff's theorem extends the validity of this result to general relativity.

Consider now a homogeneous sphere of radius R, density ρ and mass $M=(4\pi/3)\rho R^3$. A mass at its surface is given some outward radial velocity. Energy momentum conservation gives at any $r>R$ where the mass has velocity $V(r)$:

$2GM/r - V^2(r)=K$, K being a constant. From $V^2(r) = 2GM/r-K$ we see that for $K<0$ we can calculate V up to $r=\infty$ where $V=\sqrt{-K}$. The mass μ escapes the gravity of the sphere. For $K>0$ there is a distance $r_{max}= 2GM/K$ where $V=0$ and beyond which the mass falls back onto the sphere. In between, for $K=0$, the mass just escapes and, at any r, its velocity is equal to the escape velocity[11] at that r, $V_{esc}(r)=\sqrt{(2GM/r)}$.

Imagine now that the sphere expands or contracts in such a way that the mass remains always just above its surface $R=r$. This does not modify in any way the movement of the mass since the sphere acts on it as if all its mass were concentrated at its centre. But, now, the mass is at rest with respect to the surface of the sphere. The sphere radius and the sphere density now depend on time, the density like $1/R^3$. The expansion, or contraction, of the sphere preserves its homogeneity. It is convenient to think of a comoving reference frame attached to the sphere and having a unit length $a(t)$ on each of the three axes, that expands or contracts as the sphere does. A point of the sphere having fixed comoving coordinates (x,y,z) has therefore ordinary coordinates $r=(x\,a(t),\, y\,a(t),\, z\,a(t))$ and a velocity $V=(x\,da/dt,\, y\,da/dt,\, z\,da/dt)$, namely $\mathbf{V}=H(t)\mathbf{r}$ with $H(t)=da/adt$. Here $H(t)$ is called the Hubble constant (constant with respect to r, but possibly t dependent!). The distance between any two points and its time derivative are proportional to each other.

[10] This is only true in three dimensions, it happens because the forces generated by the two pieces of a spherical shell seen under a solid angle ω are equal (same ratio of the area of the shell to the distance squared, same thickness of the shell, same angle with respect to the line of sight). In one or two dimensions it is no longer true. Inside a two-dimensional disk or a one-dimensional filament, one is attracted by the boundaries of the object and one cannot escape it.

[11] Black holes correspond to the situation where the escape velocity reaches the light velocity, namely where the radius of the sphere does not exceed $R_{Schwarzschild}=2GM$, the Schwarzschild radius.

We now evaluate the forces acting on a mass μ located inside the sphere, at a distance λr from its centre, $\lambda<1$ constant with time. Energy momentum conservation reads $2GM\lambda^3/(\lambda r)-(\lambda V)^2=0=\lambda^2(2GM/r-V^2)=\lambda^2K$. Thus, for any point inside the sphere, any increase of its gravitational potential is exactly compensated by a decrease of its kinetic energy, it is therefore at rest in the comoving frame. Its K value is simply λ^2K. This implies that we do not need to keep the sphere together by some magic additional force, all points of the sphere move precisely as they should to satisfy Newton dynamics. We have therefore found the movement of a spherical homogeneous medium under its own gravity.

The last step is to let the sphere radius become very large compared to the size of the volume of the medium we wish to consider in its interior. As we may add as much material as we wish outside the sphere as long as it remains spherically symmetric, this is not a problem. The remarkable property of Hubble relation to be obeyed at any point if it is obeyed at one point will make the centre of the sphere become irrelevant in the limited space that we consider. Rewriting energy momentum conservation in terms of the time dependent (but space independent!) density $\rho(t)$ gives $(8/3)\pi G\rho r^2-V^2=K$ with $V=Hr$. As it is obeyed for any r, it is sufficient to write it down for the scale parameter $a(t)$ that describes the movement completely. Replacing da/adt by $H(t)$ and dividing by a^2 we find

$$H^2=(8/3)\pi G\rho-K/a^2 \tag{5.6a}$$

$$H=da/adt \tag{5.6b}$$

These two equations summarize completely the movement of an infinite homogeneous medium under the action of its own gravity. Here a, ρ and H are time dependent but space independent, while K is time independent but depends on the choice of scale[12]. Depending on its sign, the medium will either infinitely expand (if $K<0$) or contract after having somewhat expanded (if $K>0$). The limiting case, ($K=0$), corresponds to what is called the critical density,

$$\rho_{crit}=3H^2/8\pi G \tag{5.6c}$$

A medium having a density greater than the critical density will stop expanding and contract at some point (ending up in what is called a "big crunch") while a medium having a density inferior or equal to the critical density will expand for ever. We may rewrite (5.6a) as $K/a^2=H^2\,\rho/\rho_{crit}-H^2$, namely

[12] Indeed, there is arbitrariness in choosing the scale: we may choose it as we wish, K will simply scale with a^2. Do not be misled by this assertion: of course, K, which is time independent, does not scale as a^2 as a function of time! What it means is that choosing a scale $a'=\lambda a$, with λ constant in time, implies having $K'=\lambda^2K$.

$$K/a^2 = H^2(\Omega-1) \tag{5.6d}$$

$$\Omega = \rho/\rho_{crit} \tag{5.6e}$$

Finally we may write the so called "force equation" stating that the acceleration $dV/dt=-(4/3)\ G\ \pi r^3\rho/r^2$, namely[13]

$$d^2a/dt^2=-(4/3)\pi G\rho a \tag{5.6f}$$

All we need to do to obtain Friedmann equations is to redefine ρ as the energy density rather than the mass density, which is very natural in view of the arguments developed in chapter 3. However, Equation 5.6f (that is essentially the derivative of the 5.6a) takes a slightly different form in the case of a radiation dominated Universe. To see this let us consider a Universe made of photons. The wavelength of a given photon expands as $a(t)$, namely its frequency, or equivalently energy, decreases like $1/a(t)$, a consequence[14] of relation 5.4. The energy density, instead of decreasing as $1/a^3(t)$ as in the matter-dominated case (describing the decrease of the number of photons per unit volume), decreases now as $1/a^4(t)$ as the energy per photon decreases as $1/a(t)$. Photons are of course not at rest in the comoving frame, in fact they fly at speed of light!

Consider a volume V containing an energy ρV, say a small cylinder of length l and cross-section S. Change l by dl. The photons exert a pressure p on the cylinder, namely a force pS that makes a work $pSdl=-pdV$ and that must compensate the change in energy $dE=d(\rho V)$, namely $d(\rho V)+pdV=0$. Quite generally we have therefore $d(\rho a^3)/da=-pd(a^3)/da=-3pa^2$, namely, for photons that have $d(\rho a^4)=0$, $ad(\rho a^3)/da=-\rho a^3=-3pa^3$. Photons exert therefore a pressure $p=\rho/3$ that needs to be taken into account in the force equation.

In general by differentiating the second equation with respect to time we get: $2VdV/dt=(8/3)\pi Gd(\rho a^2)/dt$, namely $d^2a/adt^2=(4/3)\pi G(d(\rho a^2)/dt)/(ada/dt)$.
As $d(\rho a^3)/da=-3pa^2$, $ad(\rho a^2)/da+\rho a^2=-3pa^2$ and $d(\rho a^2)/dt=-(\rho+3p)(ada/dt)$.

Replacing in the former equation we get

$$d^2a/adt^2=(-4/3)\pi G(\rho+3p) \tag{5.6g}$$

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[13] There is no new information in this force equation. Indeed it is simply obtained by differentiating the energy momentum conservation relation with respect to time: $d\{(8/3)\pi G\rho a^2-(da/dt)^2\}/dt=0$, $2\ da/dt\ d^2a/dt^2=(8/3)\pi Gd(\rho a^2)/dt$ and, as $d(\rho a^3)/dt=0$, $ad(\rho a^2)/dt=-\ \rho a^2da/dt$ namely $d^2a/dt^2=-(4/3)\pi G\rho a$.

[14] Note that the relativistic expression for the Doppler shift is $1+z=\sqrt{(1+\beta)/(1-\beta)}$. But this is irrelevant here.

which is the relativistic form of the force equation. Here $p=0$ for a matter dominated Universe and $p=\rho/3$ for a radiation dominated Universe. In general the relation between p and ρ is called the equation of state and is written in the form $p=w\rho$.

The deceleration parameter is defined as

$$q = -d^2a/adt^2/H^2 = \Omega/2+3p/(2\rho_{crit}) \tag{5.6h}$$

namely a flat, matter-dominated Universe decelerates with $q=1/2$.

We may now rewrite Friedmann equations 5.6 in their general form[15]:

$$H^2(t)=(da/a(t)dt)^2=(8/3)\pi G\rho(t)-k/a^2(t) \tag{5.7a}$$
$$k/a^2(t)=H^2(t)\{\Omega(t)-1\}, \qquad \Omega(t)=\rho(t)/\rho_{crit}(t), \quad \rho_{crit}(t)=3H^2(t)/(8\pi G) \tag{5.7b}$$
$$d^2a/a(t)dt^2= -(4\pi G/3)\ (\rho(t)+3p(t)) \tag{5.7c}$$
$$q= \Omega(t)/2+3p(t)/(2\rho_{crit}(t)) \tag{5.7d}$$

The very same equations would have been obtained by going through the Einstein equations. The kindergarten treatment that was followed here provides a remarkably simple picture of what is going on. However, one should not be mislead by this simplicity and ignore the subtle difficulties, inherent to gravity, which have been pointed out earlier.

Note that the Hubble relation that describes expansion allows for relative velocities to exceed the light velocity. One may find it disturbing but we must remember that special relativity is no longer valid; it has become only a local approximation. As we shall see below the distance where this happens is called the event horizon, $R_H=1/H$. Obviously, R_H is the Schwarzschild radius of a flat Universe contained within the event horizon (the velocity being precisely equal to the escape velocity). One may say, from that point of view, that we are living in a black hole! But the concept of black hole came about in the context of a Schwarzschild metric, not of a FRW metric! In the case of a flat matter dominated Universe, as ours is today, multiplying 5.7a by R_H^3 gives $R_H=2GM_H$ where M_H is the mass contained within the horizon. The integral of the kinetic energy is $T_H=0.3M_H$. This is at the scale of the gravitational energy contained in such a Universe, $GM_H^2/R_H=0.5M_H$. One should not take this kind of oversimplified

[15] The more general form includes a cosmological constant Λ which adds to $8\pi G\rho(t)$ in 5.7a. While the presence of such a constant is favoured by presently available data, we set it to zero for the time being; we will return to that point later.

arithmetic too seriously but it shows that one should not be surprised when one finds relations of such kinds.

5.4 Evolution of a flat Universe

For a flat Universe (which is the case of ours today) $da/dt=\{(8/3)\pi G\rho\}^{1/2}a$. As ρ is inversely proportional to a^3 for a matter dominated Universe and to a^4 for a radiation dominated Universe, da/dt is inversely proportional to $a^{1/2}$ and a respectively, implying that a varies as $t^{2/3}$ and $t^{1/2}$ respectively (Figure 5.1). Correspondingly $V(t)$ varies as $t^{-1/3}$ and $t^{-1/2}$ respectively.

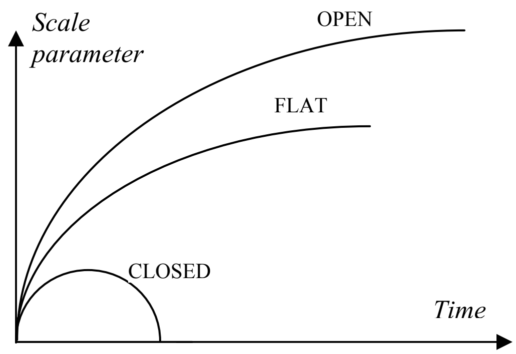

Figure 5.1Time evolution of the scale parameter

From Friedmann equation,
$k/a^2(t)=H^2(t)\{\Omega(t)-1\}$, or $k=V^2(t)\{\Omega(t)-1\}$,
we see that $\Omega(t)-1$ must decrease as fast as $V^{-2}(t)$ when t decreases and therefore Ω must become closer and closer to 1. In fact, as we know that today Ω is very close to 1, we can infer that at GUT time[16] Ω must equal 1 to an accuracy of 49 decimals! This is sometime referred to as the "flatness problem" because in models (claiming of course to describe the initial conditions at GUT times) that do not imply from first principles that Ω is exactly 1 it would be very difficult to fine tune it to such a precision.

The table below summarizes the evolution of the main parameters in the Universe. The last entry, temperature, deserves some explanation.

Parameter	Matter dominated	Radiation dominated
Pressure p	0	$\rho/3$
Energy density ρ	t^{-2}	t^{-2}
Expansion scale a	$t^{2/3}$	$t^{1/2}$
Expansion velocity V	$t^{-1/3}$	$t^{-1/2}$
Hubble constant H	t^{-1}	t^{-1}
Temperature	$t^{-2/3}$	$t^{-1/2}$

[16] Grand Unification Theories (GUT) try to unify the strong force and the electroweak force, both of which are effective forces described in the standard model of elementary particles, in a unified way implying a group symmetry higher than SU(3)×SU(2)×U(1). From the extrapolation of the effective coupling constants it is possible to infer that such unification should occur in an energy range at the level of a percent of the Planck mass.

We know that the cosmic microwave background has a black body spectrum with an energy density *(E=hv)*

$d\rho/dv = 8\pi hv^3/(\exp[hv/k_BT]-1)$, or, writing $x= hv/k_BT$,

$d\rho/dx = 8\pi hx^3(k_BT/h)^4/(e^x-1)$. As ρ is proportional to T^4 and to a^{-4}, the temperature is therefore inversely proportional to a, namely proportional to $t^{-1/2}$and $t^{-2/3}$ in a radiation dominated and matter dominated Universe respectively.

Note that in both matter and radiation dominated era ρt^3 is proportional to t: the mass of the Universe contained in the horizon scales with the age of the Universe. Knowing how the parameters of the (flat) Universe evolve with time makes it possible to infer the age of the Universe, t_0, from the knowledge of its present state, $H_0=H(t_0)$. Neglecting the radiation dominated period we have:

$a(t)=a(t_0)(t/ t_0)^{2/3}$, $da/dt=(2/3)a(t_0)(t/ t_0)^{-1/3} /t_0 =(2/3)a(t)/t$

Namely, $H(t)=da/adt=2/3t$ and $t_0=2T_0/3 = 2/(3H_0)$ where we have introduced the so-called "Hubble age", $T_0=a(t_0)/(da(t_0)/dt)=1/H_0$.

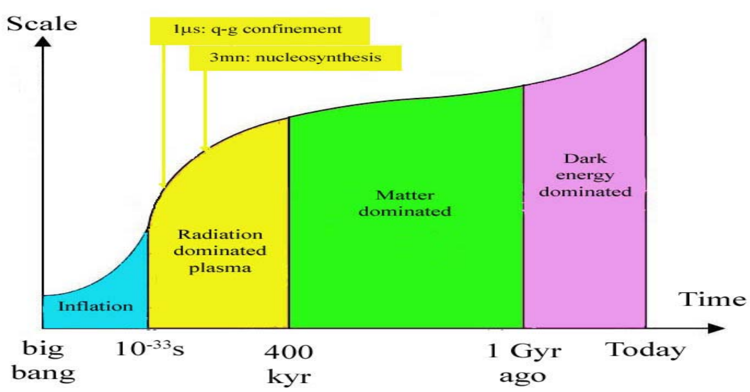

Figure 5.2

A schematic illustration of the evolution of the Universe

Plugging in the currently accepted value of the Hubble constant, 71 km/s/Mpc, gives an age of only 9.2 billion years. In fact the best current estimate of t_{now} is 13.7±0.2 billion years (WMAP), as obtained from a more realistic account of what the Universe is made of (including dark energy, see below). It happens to be nearly equal to T_0.

5.5 Horizons and the causality problem

For two points to be causally connected at time t their distance $d(t)$ must not exceed the "horizon" defined as the distance spanned by light during that time, namely t $(c=1)$. But for some time $t'<t$ these two points must have been causally disconnected. In the matter dominated era it happened when

$d(t')=(t'/t)^{2/3}d(t)=t'$, or $d(t)/t=(t'/t)^{1/3}$, $t'=t(d(t)/t)^3$

Similarly in the radiation dominated era we get $t'=t(d(t)/t)^2$

This implies that the sky that we see today was in fact causally disconnected at some earlier time. Then how can it be that it is so homogeneous? Even if two causally disconnected regions can always be linked by a chain of causally connected regions, one should be bothered by this situation that is usually referred to as the "causality problem" or "horizon problem". Rather than thinking of causally disconnected regions, one should rather think of the relative gradient describing the space variation of some quantity, say the temperature T of a plasma in thermal equilibrium: dT/Tdl. It is this quantity that is expected to be constrained to be $\geq 1/\Lambda$, Λ being the horizon.

The horizon we are talking about here is sometimes called the *particle horizon*. One also refers in other circumstances to what is called the *event horizon*. It is defined as the distance where the Hubble velocity reaches the light velocity, that is precisely the Hubble time T_0. As we have seen, in the case of our present Universe, both horizons are nearly identical.

5.6 Inflation

In addition to the causality and flatness problems, the standard big bang model must face two more difficulties: the absence of monopoles in today's Universe and the very large value of the product ρa^4 during the early radiation dominated era. The first problem stems from the belief that at GUT times there must have been stable magnetic monopoles in such abundance that some should have been observed today in the very sensitive searches that have been performed.

The second problem is that ρa^4 is a constant during the radiation dominated era that should be obtained from first principles. But in natural units ($\hbar=c=1$), it is a pure number and one would therefore expect its value to be commensurate with unity. However a lower limit of this quantity is obtained by considering photons only (the density of which is well known), yielding the result $\rho a^4>10^{115}$. This is far from unity and may discourage theorists to devise a sensible model...

It is true that none of these problems may sound dramatic when one remembers that anyhow theory must fail at the Planck mass ($\sim10^{19}$ GeV) that is only 2 to 3 orders of magnitude above the GUT mass ($>10^{16}$ GeV) and that we do not really know precisely what we are talking about in this region. Yet, there exists a model, the inflation scenario, which disposes simply of all these problems and has therefore become popular, indeed gaining credibility with time as its predictions were better and better verified by observation. It has emerged as a very sensible working hypothesis and is now included in the standard model of modern big bang cosmology. Yet, given its very conjectural nature, a hand waving presentation of its main features will be sufficient in the present introduction.

Let us assume that at GUT times the Universe has been for a while in a meta-stable state, similar to that produced by the potential invoked to describe the Higgs mechanism. In such a state a volume V embeds an energy $\rho_{meta}V$, where ρ_{meta} is the constant energy density associated with that state. We neglect any other form of energy density, matter or radiation. Increasing V by dV simply increases that energy by $dE=\rho_{meta}dV$, the real vacuum being taken as having zero energy density. As $dE=-pdV$ this implies a uniform negative pressure, $p=-\rho_{meta}$. From Friedmann force equation $d^2a/adt^2=-4\pi G/3\ (\rho+3p)$ we get $d^2a/adt^2= 8/3\ \pi G\ \rho_{meta}$ which can be easily integrated as $a=exp(Ht)$ with a really constant (both in time and in space) Hubble constant, $H=\sqrt{(8/3\ \pi G\ \rho_{meta})}$.

As ρ_{meta}, in natural units, has dimension of $(mass)^4$ and as the only scale at our disposal is the GUT scale $(>10^{16}\ GeV)$ we expect ρ_{meta} to be of order $10^{64}\ GeV^4$. Putting numbers in gives a value $H{\sim}10^{36}s^{-1}$ for the Hubble constant. The inflation scenario assumes that the Universe was in such a regime from the big bang up to GUT time, $t_{GUT}{\sim}10^{-33}s$. Hence $Ht_{GUT}{\sim}\ 10^3$.

During inflation ρa^4 blew up by a factor $exp(4Ht_{GUT})$ that we should like to be of the order of 10^{115} for the density to join smoothly to its expression in the subsequent radiation dominated era. This means that we should like Ht_{GUT} to be of the order of $0.25\times115\times ln10\ =\ 66$. This is easily achieved by choosing $\rho_{meta}{}^{1/4}=\sqrt{.066}\ 10^{16}\ GeV=\ 1/4\ 10^{16}\ GeV$. The problem of the large ρa^4 value is therefore solved.

The flatness problem is similarly trivially solved : whatever was the value of Ω before inflation, it has been driven very rapidly to 1 during inflation where $H^2(\Omega-1)$, and therefore $(\Omega-1)$ itself, have decreased by a factor $exp(-2Ht_{GUT})$ of order 10^{58}.

The monopole problem also is solved because any primordial particle has been diluted to such an extent during inflation that the probability to detect it today is negligibly small.

Last, let us consider the causality problem. What happened during inflation is that a small causally connected region of the Universe has been suddenly blown up to our presently observable Universe, solving the causality problem. The Universe that we see today may be but a very small fraction indeed of the whole Universe (whatever that means).

The inflation scenario has the further attraction of explaining the inhomogeneities observed today, galaxies, clusters, etc... as resulting from quantum fluctuations of the inflation field that have been blown up by a factor 10^{29} during inflation, leading to the density fluctuations that acted as seeds for the gravitational condensation of matter into galaxies.

6. *Our Universe*

6.1 A non homogeneous Universe

Our Universe is far from being homogeneous: most of its non relativistic energy is concentrated in galaxies. Does this prevent the use of a FRW metric and does it invalidate the conclusions of the preceding chapter?

In 1945, Einstein and Strauss presented arguments showing that we could still use the FRW metric as long as the distribution of galaxies in the Universe is homogeneous on a large scale. This is often referred to as the "Swiss cheese model" of the Universe. The argument goes as follows:

Consider a point mass M, say a star or galaxy, or even a cluster of galaxies, surrounded by vacuum and very distant from any other matter and assume that this other distant matter can be described as being a homogeneous Universe of density ρ_e. Consider a spherical bubble centered on M (Figure 6.1) and having a radius R_b such that $4\pi R_b^3/3=M/\rho_e$. Another way to look at it is to start with a perfectly homogeneous Universe and to condense all the matter contained in a sphere of radius R_b at the centre of the sphere. Inside the bubble one would like the Universe not to expand: systems that are gravitationally bound should stay so when the Universe expands. What we want inside the bubble is a Schwarzschild metric, which corresponds to a static, namely not expanding, Universe. Is it possible to have a Universe with such a hybrid metric, Schwarzschild inside the bubble and FRW outside? Einstein and Strauss have shown that the answer is yes. Our kindergarten treatment of Friedmann equations makes this result kind of obvious: all one needs to do is to keep the bubble growing as the Universe expands (or having a fixed radius in the comoving frame); we would not learn more physics by going through the calculation and I shall skip it. It is then easy to see that, by simply repeating the same game around each galaxy, or cluster of galaxy, we can describe a Universe equipped with a static Schwarzschild metric nearly everywhere where there are galaxies and retaining the FRW expansion to simply scale up the distances between these galaxies.

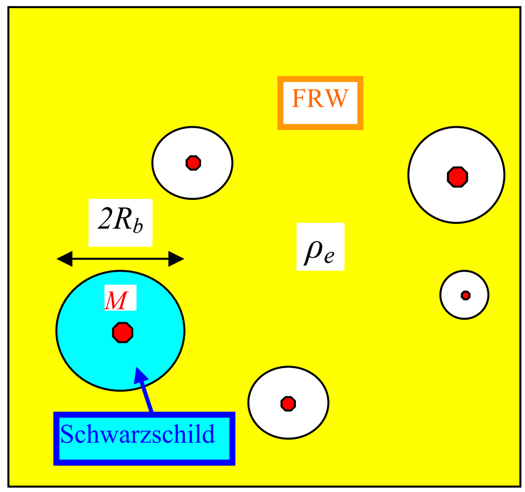

Figure 6.1 The Swiss cheese model of Einstein and Strauss

This, however, may bring up more questions and problems than it is really answering and solving. Which prescription should we use to draw the bubbles?

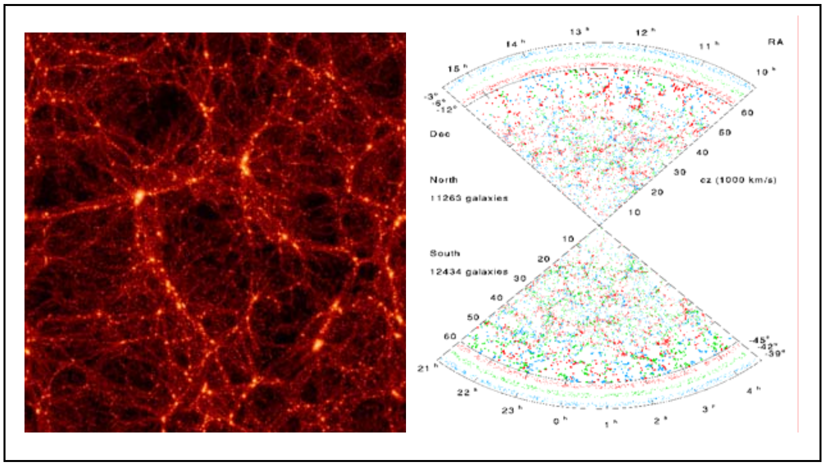

Figure 6.2 The wall and void structure of the Universe at large scales showing a simulation (left) compared to real data (right)

Who decides where FRW applies and where Schwarzschild applies? More importantly, we know today that galaxies and clusters of galaxies are not at all distributed homogeneously in the Universe (Figure 6.2) but on two dimensional surfaces (the walls) that intersect in one dimensional lines (where major clusters of galaxies are found) and embed three dimensional volumes of emptiness (the voids). How can such a structure expand? A simple answer that comes to mind is to think of the two dimensional surfaces to expand, with the three dimensional empty spaces consequently expanding *à la* Hubble, namely the walls, as surfaces, being fixed in the comoving frame. But this would mean that the wall thicknesses do not grow with the expansion, contrary to Hubble relation. If this simple idea had anything to do with reality, a Swiss cheese metric would surely not be what we need, none of the assumptions that were made would apply here. We do know that this "walls and voids" structure has appeared with time rather than getting diluted: for the same reason as for a cluster of galaxies it seems that we should like a static Schwarzschild metric to govern the formation of a wall in the direction normal to the wall (to prevent it from expanding). But in the wall plane, if we do not accept having expansion, how can we afford having any expansion at all? The kind of hybrid metric one would then need would be Schwarzschild normal to the walls and expanding in the wall planes. It seems then that our arguments would have to be revised in depth and that Friedmann equations would no longer be valid. I am fully aware of the fact that such comments are simply bringing confusion in the mind of the students, but I hope that they may help in thinking critically and in some depth about these problems. There is no point to go through the detailed tensor arithmetic of FRW if one does not know how to answer such basic questions. Indeed, these issues, today, are quite controversial and some authors claim that dark energy could simply be the result of accepting too blindly that FRW applies to our Universe today. An argument (weak I must say) that they bring about is that dark energy dominance occurred more or less at the same time when matter condensed into structures.

6.2 Dark energy and gravity at large distances

Taking seriously the evidence provided by the analysis of the angular power spectrum of the CMB that our Universe is flat, implies, as we have seen in the first chapter, that some 73% of its energy density is not accounted for by matter and radiation, whether visible or dark, baryonic or else. This surprising conclusion has received support from other observations, in particular from the fact that very distant galaxies appear fainter than they should, thereby suggesting that they are farther away than we estimate, namely that the expansion of the Universe is currently accelerating rather than decelerating. The "deceleration" parameter measured in this way takes the value $q=-0.67\pm 0.25$.

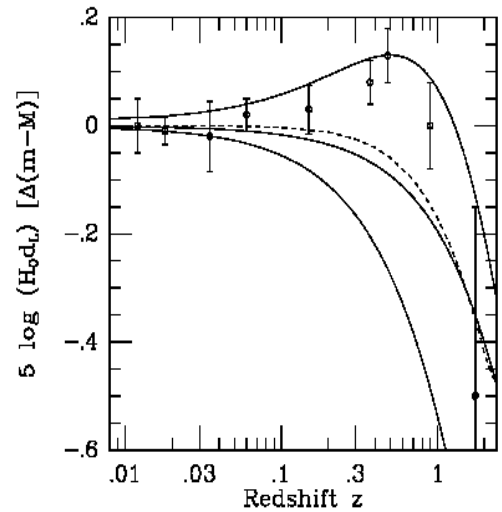

Figure 6.3 Evidence for the fainter appearance of very distant galaxies.

For a flat Universe expanding as a power law, as we saw in Section 5.4 :

$a(t)=a_0 t^n$, $da/dt=na_0 t^{n-1}$, $H=n/t$, $d^2a/dt^2=n(n-1)\, a_0\, t^{n-2}$ and $q=-(n-1)/n$.

While for a Universe expanding exponentially, as is the case for inflation and for a cosmological constant,

$a(t)=a_0 e^{\lambda t}$, $da/dt= a_0 \lambda e^{\lambda t}$, $H=\lambda$, $d^2a/dt^2= a_0 \lambda^2 e^{\lambda t}$ and $q=-1$.

As, from Friedmann's force equation 5.7d, $q=(1+3w)/2$, we find $w=-1+2/3n$ in the power law case and $w=-1$ in the exponential case. These results are summarized in the table below:

	w	*q*	*n*
Matter dominated	*0*	*1/2*	*2/3*
Radiation dominated	*1/3*	*1*	*1/2*
Inflation; dark energy	*−1*	*−1*	∞

The WMAP global analysis gives $w= -0.93\pm 0.06$ while from $q=-0.67\pm0.25$ quoted above we get $w=-0.78\pm0.16$. The dark energy "equation of state" is therefore found consistent with that given by a cosmological constant Λ, $w=-1$. While not much else is known of this so-called "dark energy", it is, together with inflation, the recent addition to the standard big bang model that makes it become the current standard model of cosmology, often called the "concordance model". It is consensually used by most cosmologists and astrophysicists as a convenient phenomenology to describe the evolution of the Universe on the road to a better understanding of its nature.

A cosmological constant would describe a repulsive force that would increase linearly with distance as Λr. It was first introduced by Einstein in order to allow for a static solution of his equations applied to a homogeneous Universe, which is not possible otherwise (at that time one did not know about the expansion of the Universe, it was before Hubble's work). Adding a cosmological constant is like adding a constant term to the energy density, $\rho_\Lambda=\Lambda/8\pi G$ corresponding to a constant negative pressure, $p_\Lambda= -\rho_\Lambda$. Namely work has to be done on the Universe in order to contract it: it corresponds to a repulsive potential which increases with distance in proportion to its square and its effects are therefore the more important the larger the distance. As the Universe seems to have become dark energy dominated only recently, we see that the problem of understanding dark energy is a problem of understanding gravity at very large distances.

The inclusion of a cosmological constant in Einstein equations, while perfectly acceptable from a mathematical view point, is not very satisfactory from a physics point of view because what is required to describe dark energy is something like 10^{-120} in natural Planck units instead of something commensurate with unity if the it had anything to do with the zero point energy of vacuum! Which physics would it be hiding then? This is why, despite its success at describing the Universe as we know it today, it is usually not accepted as a welcome answer to the problem. Whatever will ultimately be accepted as a satisfactory solution, it is clear that we should be prepared to major revisions of our understanding of gravity above Gpc distances.

6.3 Quantization of gravity

A well known limitation of general relativity, or for that matter of any macroscopic theory of gravity, occurs at microscopic scales. Consider a wave packet having an energy spread ΔE and dimension $R=\Delta t=1/\Delta E$ (such that they satisfy Heisenberg uncertainty relations). The maximum value that the gravitational energy, $\sim GM^2/R$, can take is ΔE, in which case $GM^2=1$. For masses higher than the Planck mass, $M_{Planck}=1/\sqrt{G} \sim 10^{19}$ GeV, the gravitational energy exceeds the energy spread of the wave packet and our theories must therefore be revisited. The length and time associated with the Planck mass are $1/M_{Planck}=\sqrt{G}$ namely a Planck length of 2 10^{-33} cm and a Planck time of 6 10^{-44} s. Note that the Schwarzschild radius of the Planck mass is $2M_{Planck}G=2\sqrt{G}$ namely twice the Planck time: like the visible universe and the horizons of various black holes, the Planck regime stands on the Schwarzschild line of Figure 1.9. Indeed, setting $G=1$ to extend our definition of natural units, the three Planck quantities, mass, length and time are equal to unity.

Supersymmetry, a fundamental symmetry that transforms bosons into fermions and goes a long way toward the resolution of the problems related to the generation of masses in particle theory, has open the way to a possible quantization of gravity. Indeed, its fundamental commutators imply the momentum operator and gauging the theory generates a gravity field. However, such supergravity theories are not renormalizable. Their expression in the framework of string theories, on the contrary, seems to be not only renormalizable but even finite. Strings live at the Planck scale in a high dimensionality space à la Kaluza-Klein, all space dimensions but three being compactified, namely curled onto themselves, again at the Planck scale. Superstrings seem to generate this way a ten-dimensional quantum gravity. If these views were confirmed – we are still a long way away from that state – general relativity would not need to be quantized: it would indeed be the macroscopic effective expression of a quantum theory.

6.4 Some tests of general relativity

Possible deviations from general relativity are measured from a systematic global analysis of all existing relevant measurements. They are summarized by only two parameters, which compare the measurements to the predictions of general relativity. One is for the difference between the Euclidian metric and the general relativity metric, the other for non-linearities of general relativity gravity. They are found not to exceed 2 10^{-3} and 10^{-3} respectively. The most accurate radio measurement of the deviation of photons by the Sun yields an accuracy of only 1% on the first of these two parameters. The three most constraining measurements use laser reflectors on the Moon, radar echoes from the Viking station on Mars and a global analysis of the dynamics of the solar system including, in particular, the advance of the planetary perihelions.

There is today overwhelming evidence that black holes are present in the Universe, both stellar black holes having typical masses of a few solar masses and galactic black holes. The masses of the latter cover a very broad range. That at the centre of the Milky Way is the object of intense studies and has a mass of 30 million solar masses. A series of pictures is shown in the appendix, summarizing our knowledge of its properties. They make up an amazing collection of recent data of outstanding quality illustrating in a spectacular way the progress made: it is not so long ago that many astrophysicists doubted that black holes existed. The black holes at the center of the most active quasars may reach up to 100 billion solar masses. However, black hole studies do not imply accurate tests of general relativity. Black holes manifest themselves by events occurring just outside their horizon, which in most cases are not different from what would happen just outside any compact heavy object. Any sensible model of gravity would predict that stars must collapse into white dwarfs and neutron stars. Most properties of black holes that are specific to general relativity are difficult, when not impossible, to access experimentally as they occur beyond the horizon of the black hole[17] and are therefore kind of censored to our eyes.

The prediction that there should exist gravitational waves produced in the event of a very rapid and intense modification of gravity has defeated all direct attempts to detect them experimentally. However binary pulsars, such as PSR 1913+16, discovered by Russell A. Hulse and Joseph H. Taylor in 1974, are laboratories that have offered extremely accurate indirect checks of the existence of gravitational waves. Binaries made of a pulsar and of a very dense companion, neutron star, pulsar or black hole, are very compact and imply very intense gravity fields. Following their movement over several years makes it possible to reveal changes that are specific to general relativity. Some of these are related to the emission of gravity waves, such as the advance of the periastron, the gravitational slowing down of the pulsar rotation (on itself) and the decrease of the orbital frequency. These measurements have provided a test of general relativity to the level of 3.5 parts in thousand. Other binary pulsars, such as PSR 1534+12 discovered in 1991 by Aleksander Wolszczan, have refined these measurements.

To conclude, our current ideas on gravity, be it at Gpc distances or at the Planck scale, are not satisfactory. General relativity, in many respects, has been tested to a good precision (nothing, however, in comparison to tests existing on special relativity or quantum mechanics). Much more fragile are the models that we have been using to apply it to the Universe at large distances. As was already said, a superstring description of the world at the Planck scale might shed light on the very large scale behavior of gravity, in which case general relativity would simply appear as a macroscopic approximation of the theory. For the time being, this is science fiction, but if it became true, it would be an unprecedented success:

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[17] The emission of gravitational waves during the collapse of a star is, to some extent, a test specific of general relativity but it is too small to be detectable. Gravitational waves emitted in more violent events, such as the merger of two black holes and other similar processes responsible for the most violent gamma ray bursts may be detectable. Some of the properties related to the angular momentum of black holes (Kerr black holes) may also be considered as providing tests specific of general relativity.

between the Planck scale and the size of today's horizon there are nearly 60 orders of magnitude!

Many non-orthodox approaches exist, which I have simply ignored: they do not fit in the elementary presentation of the present lectures. They include, in particular, the possibility that the constants of physics, $\hbar$, c and G, would slowly evolve with time. There exist numerous evidences against such a possibility. Particularly elegant are the studies of paleochroic haloes made by Wilkinson. These are spherical shells induced by the Bragg path of decay alpha particles emitted by radioactive inclusions in rocks. One should also mention three decades of lunar ranging measurements using a laser reflector on the Moon, placing a limit of 10^{-11} per year on the relative change of G. Other non orthodox considerations concern possible topological complications in the structure of the Universe and, more importantly, possible modifications to Newtonian mechanics (one talks of MOND, MOdified Newtonian Dynamics).

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On chapter 4

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Appendix: A collection of pictures of Sagittarius A, the black hole at the center of the Milky Way*

The series of pictures that follows illustrate our current knowledge of Sgr A*. They are given here for entertainment, as a response to the fascination which black holes exert on our minds and as a demonstration of the extraordinary progress made by astrophysics in recent years. However, as was said in the lectures, they do not teach us much on the major questions cosmology is confronting us with. This is why I have chosen this mode of presentation.

Radio and microwave observations

1. The presence of a radiosource at the centre of the Milky Way was first noticed by Jansky in 1932 and later resolved, in 1974, at the Green Bank radio telescope.
2. In the visible there is nothing to be seen.
3. Broad 90cm VLA view of the central region.
4. Zooming in at 20cm wave length.

5,6. Zooming in more at 6 cm wave length. Many supernova remnants have been identified. Star density is one million times higher than near the Sun. Many SN explosions, dense star forming region. A three arm structure is revealed in Sgr A.

7. Highest resolution VLA image, 2ly×2ly.
8. A ring of dust (6.5ly radius) fed by dense clouds 25 to 50 ly away and three arms of hot gas (>10000K) spiralling toward SgrA*. From VLBA one learns that the diameter of the source is less than 5 light minutes, that the velocity with respect to the Milky Way is very low: SgrA* is "anchored" at its centre.
9. COBE microwave view of the central region of the Milky Way.

Infrared observations

10. Already in the near infrared, one starts to see a glow.

11, 12. Zooming in (mid-infrared).

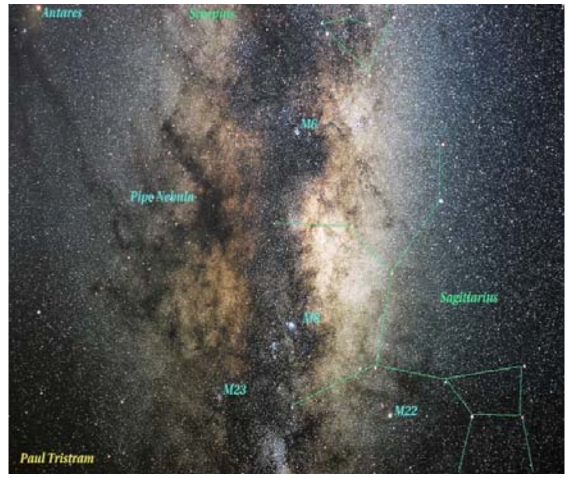
13, 14, 15. Stars are observed in infrared as orbiting around a 3million solar masses black hole.

16. High resolution infrared picture showing a faint flaring Sgr A*.

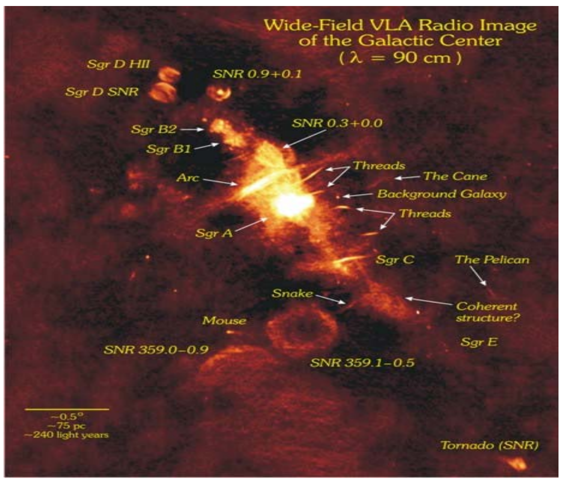
X ray observations

17. CHANDRA, used to observe Sgr A* in X rays.
18. An X ray view of the galactic centre.
19. Superposition of a 2-8 keV Chandra picture with Naos Conica VLT mid infrared. Sgr A* source <1.4 arcsec in diameter, consistent with expected accretion disc of a 3 million solar masses black hole.

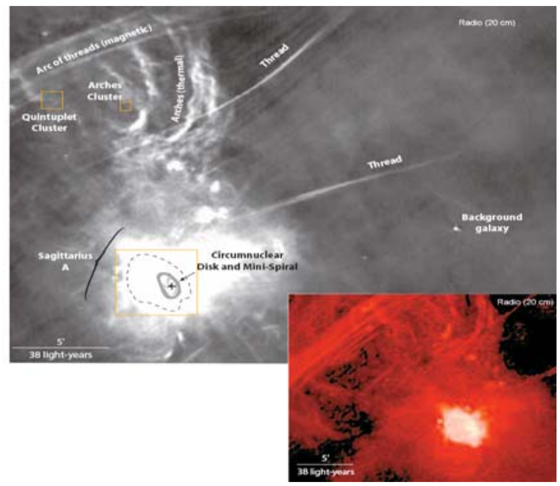
20. Observation of X ray flares.
21, 22. Possible evidence for a jet normal to the galactic plane.
23. Twenty times as many active X ray binaries as expected, suggesting that ten thousand stellar black holes may be orbiting Sgr A*.
24. Four degree gamma ray (HESS) view of the galactic centre dominated by a SNR and SGR A*. Once these are subtracted, other sources remain, one of which has apparently no counterpart.



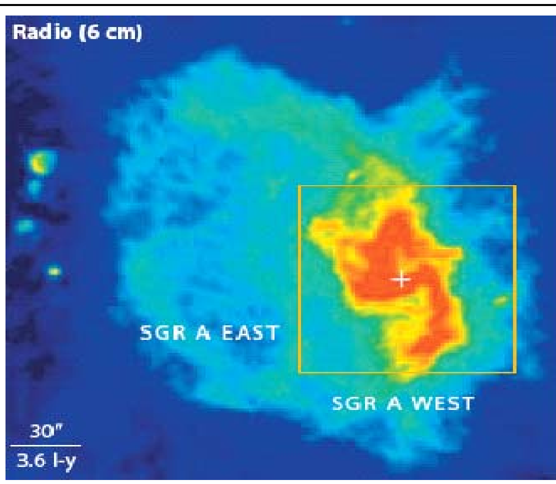
Antares
M6
Pipe Nebula
Sagittarius
M8
M23
M22
Paul Tristram



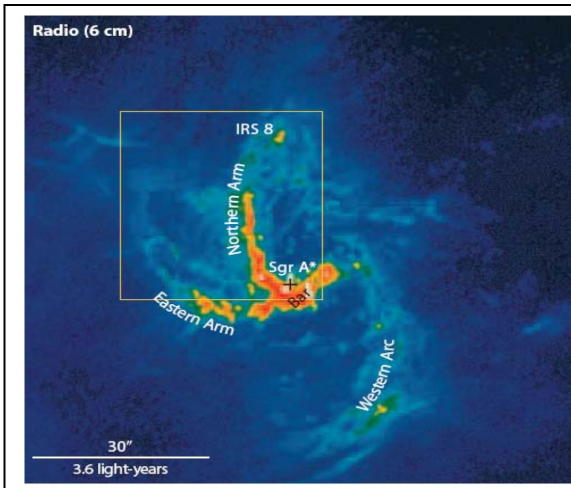
Wide-Field VLA Radio Image
of the Galactic Center
(λ = 90 cm)
Sgr D HII
SNR 0.9+0.1
Sgr D SNR
Sgr B2
SNR 0.3+0.0
Sgr B1
Threads
Arc
The Cane
Background Galaxy
Threads
Sgr A
Sgr C
The Pelican
Snake
Coherent structure?
Mouse
Sgr E
SNR 359.0−0.9
SNR 359.1−0.5
~0.5°
~75 pc
~240 light years
Tornado (SNR)



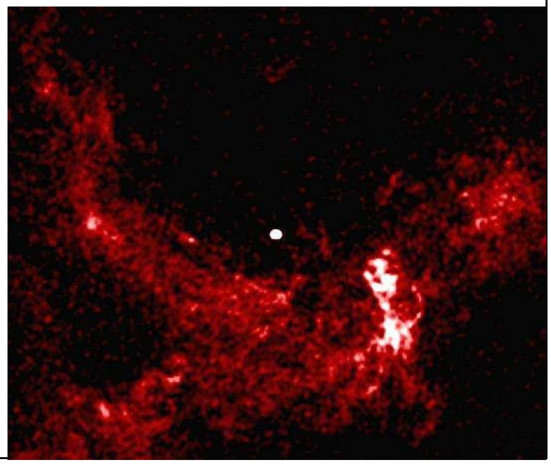
Radio (20 cm)
Arc of threads (magnetic)
Thread
Arches Cluster
Arches (thermal)
Quintuplet Cluster
Thread
Background galaxy
Sagittarius A
Circumnuclear Disk and Mini-Spiral
5'
38 light-years
Radio (20 cm)
5'
38 light-years

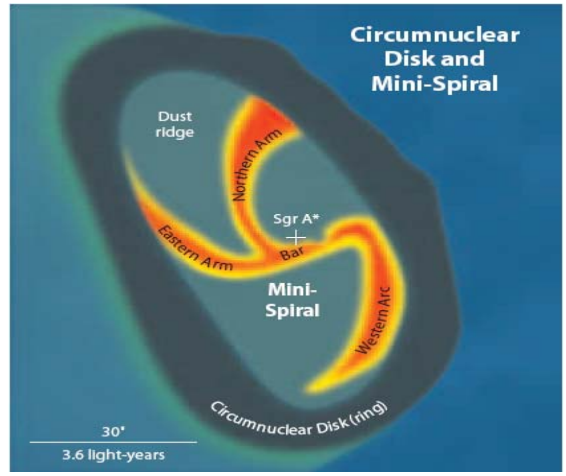


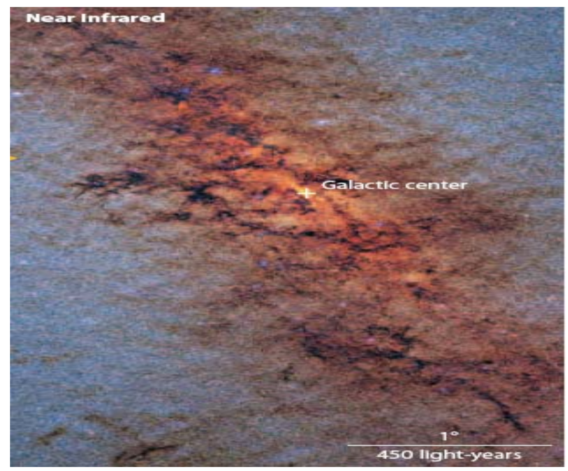
Radio (6 cm)
SGR A EAST
SGR A WEST
30"
3.6 l-y

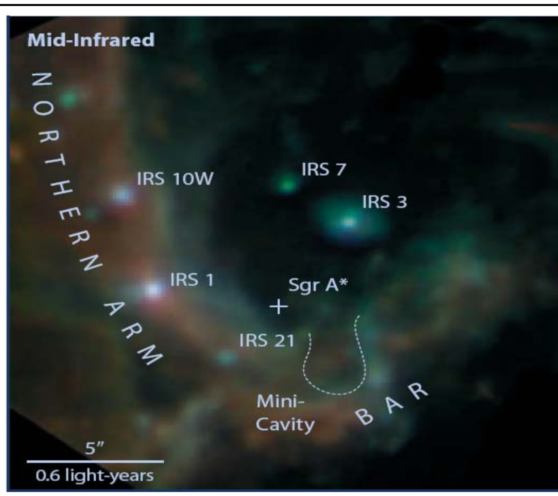


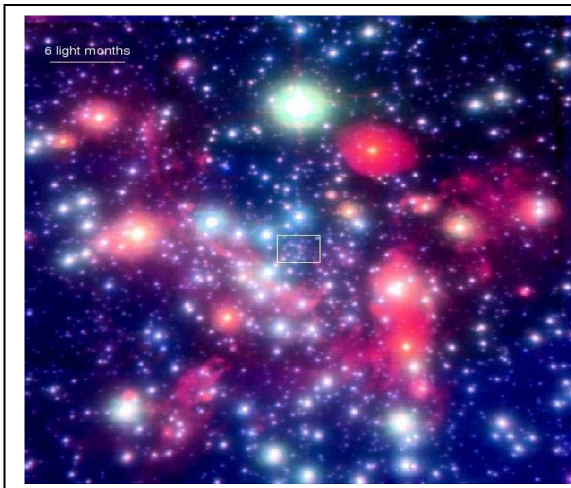
Radio (6 cm)
IRS 8
Northern Arm
Sgr A*
Bar
Eastern Arm
Western Arc
30"
3.6 light-years

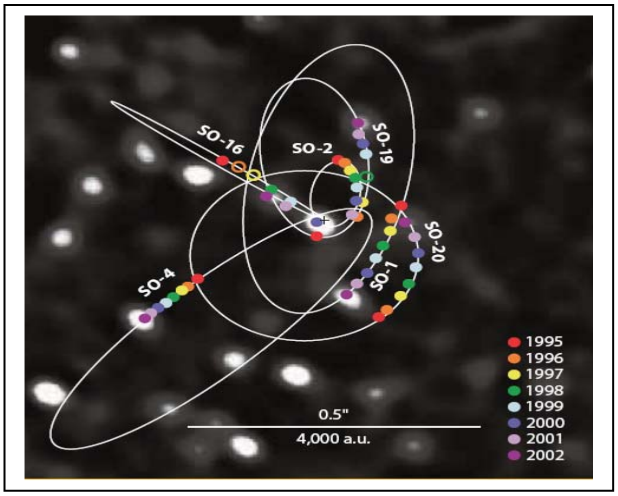


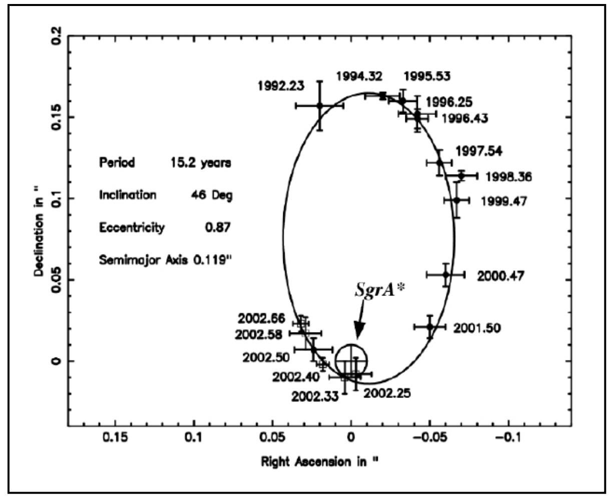

Circumnuclear
Disk and
Mini-Spiral
Dust
ridge
Northern Arm
Sgr A*
Eastern Arm
Bar
Mini-
Spiral
Western Arc
Circumnuclear Disk (ring)
30"
3.6 light-years

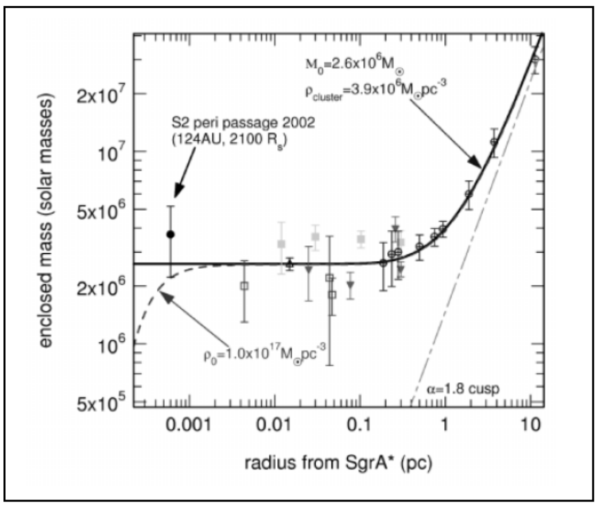

Near Infrared
Galactic center
1°
450 light-years

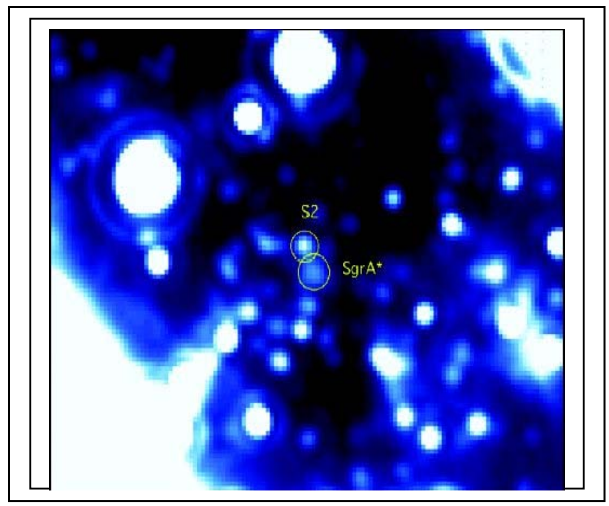

Mid-Infrared
NORTHERN ARM
IRS 10W
IRS 7
IRS 3
IRS 1
Sgr A*
IRS 21
Mini-
Cavity
BAR
5"
0.6 light-years


6 light months

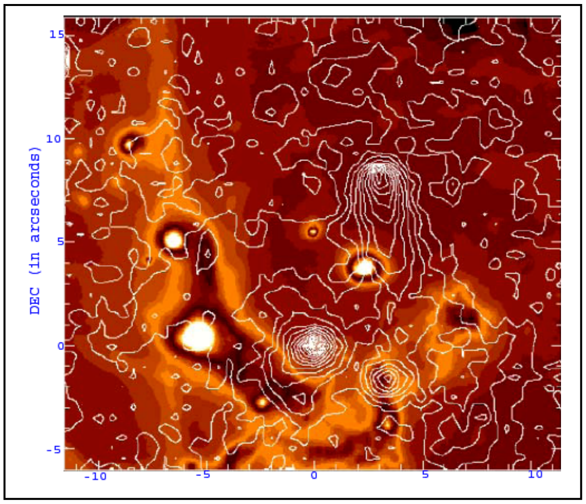

SO-16
SO-2
SO-19
SO-20
SO-1
SO-4
0.5"
4,000 a.u.
1995
1996
1997
1998
1999
2000
2001
2002

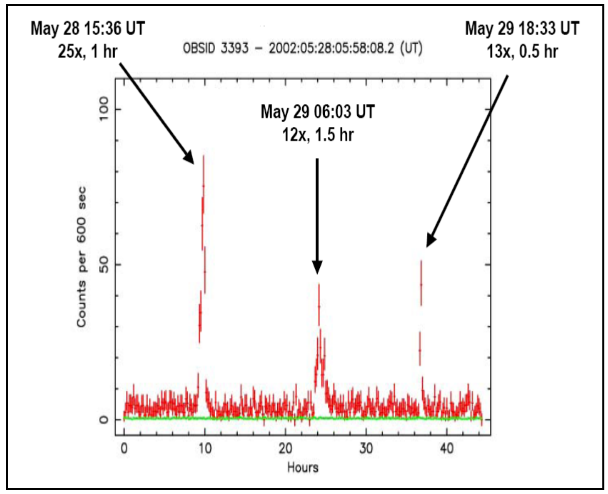

Period 15.2 years
Inclination 46 Deg
Eccentricity 0.87
Semimajor Axis 0.119"
SgrA*
1992.23
1994.32
1995.53
1996.25
1996.43
1997.54
1998.36
1999.47
2000.47
2001.50
2002.66
2002.58
2002.50
2002.40
2002.33
2002.25
Declination in "
Right Ascension in "

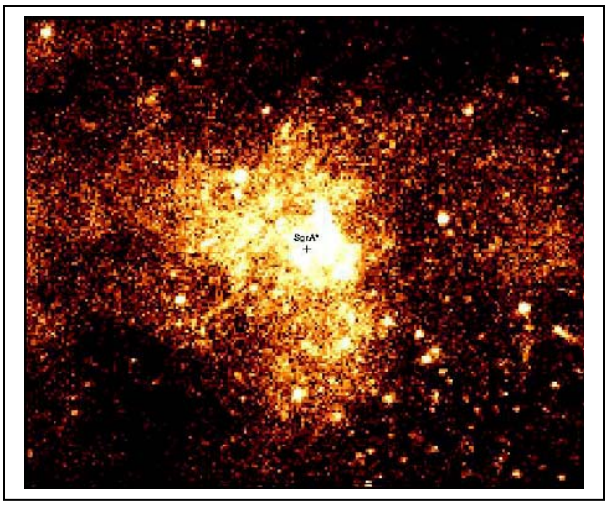

M_0=2.6x10^6$M_\odot$
$\rho_{cluster}$=3.9x10^6$M_\odot$pc^{-3}
S2 peri passage 2002
(124AU, 2100 R_s)
ρ_0=1.0x10^{17}$M_\odot$pc^{-3}
α=1.8 cusp
enclosed mass (solar masses)
radius from SgrA* (pc)

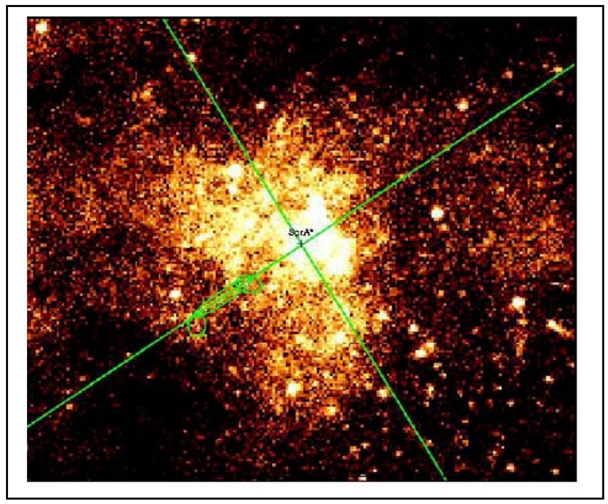

S2
SgrA*

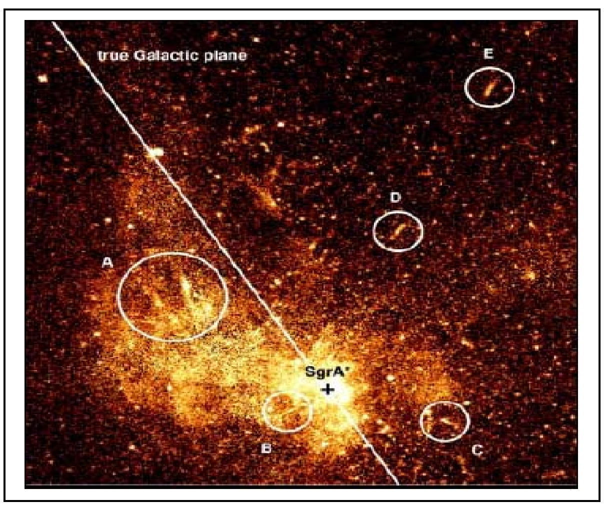

SGR A*

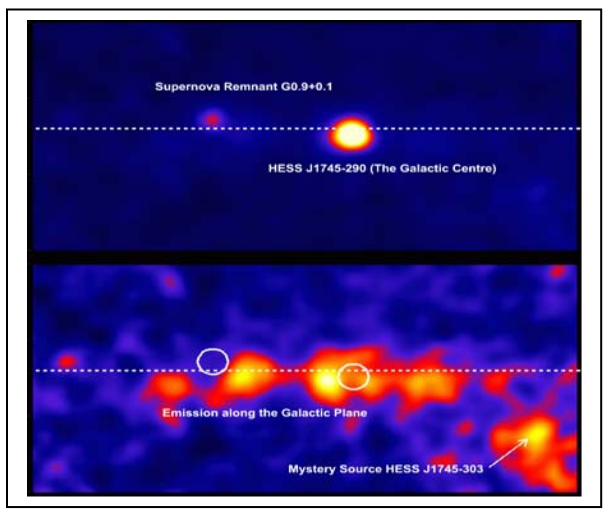

DEC (in arcseconds)
15
10
5
0
-5
-10
-5
0
5
10


May 28 15:36 UT
25x, 1 hr
OBSID 3393 – 2002:05:28:05:58:08.2 (UT)
May 29 18:33 UT
13x, 0.5 hr
May 29 06:03 UT
12x, 1.5 hr
Counts per 600 sec
Hours
0
50
100
0
10
20
30
40


SgrA*


SgrA*


true Galactic plane
E
D
A
SgrA*
B
C


Supernova Remnant G0.9+0.1
HESS J1745-290 (The Galactic Centre)
Emission along the Galactic Plane
Mystery Source HESS J1745-303